

Propositions and their Converses



0905CH09

Consider the following statement:

Statement 1: *If two sides of a triangle are equal, then the angles opposite the equal sides are equal.*

Such a statement that is either true or false is called a **proposition**.

Now consider the following proposition:

Statement 2: *If two angles of a triangle are equal, then the sides opposite the equal angles have equal lengths.*

The first proposition is of the form 'if X then Y'. The second is of the form 'if Y then X'. We say that the second proposition is the **converse** of the first proposition.

The statement 'if X then Y' is also written as 'X **implies** Y'. We will use both formulations interchangeably.

If a proposition is true, then is its converse always true?

Think and Reflect

We have proved the first statement in an earlier grade. Is the second statement true? Can you prove it?

(Hint: Draw the altitude from the vertex containing the third angle.)

Let us consider some examples of propositions and their converses,

Example 1:

- Proposition P: If it rains, then the road is wet.
- Converse Q: If the road is wet, then it has rained.

Proposition P is true but its converse, Q, may not be true. For instance, the road may be wet because a tanker spilled water on the road!

A situation that illustrates why a given proposition is not necessarily true is called a **counterexample**. It refers to an example that contradicts the stated proposition.

Finding a suitable counterexample is an important ingredient of mathematics because it allows us to show in a short and convincing way that a proposition is false.

Here is a famous example. Fermat claimed that all numbers of the form $2^{2^n} + 1$ are prime ($n = 0, 1, 2, 3, \dots$); e.g., the number $2^{2^3} + 1 = 257$. But Euler disproved this by showing that the number $2^{2^5} + 1$ is composite.

Example 2:

- Proposition P: If a number is a multiple of 6, then it is a multiple of 3.
- Converse Q: If a number is a multiple of 3, then it is a multiple of 6.

Proposition P is true. Justify why this is so.

Does this now mean that all multiples of 3 are also multiples of 6? This corresponds to the converse statement Q. Determine if it is true and if not, give a counterexample.

Example 3: In this example, n is any positive integer.

- Proposition P: If n is a perfect square, then it has an odd number of factors.
- Converse Q: If n has an odd number of factors, then it is a perfect square.

You may recall that we came across these statements in the previous grade. Which of them are true?

Discussion

Consider the following argument.

Each factor of a number has a 'partner' factor such that their product yields the given number, e.g., 5 is a factor of 35, and $5 \times 7 = 35$. Here, 7 is the partner factor of 5, and vice-versa. Let us focus on factor-partner pairs of numbers, e.g., (1, 12), (2, 6), (3, 4) are the pairs for the number 12.

If a number has an odd number of factors, there must be a factor-partner pair in which the same number repeats (e.g., the partner factor of 5 in 25). If not, the given number will have an even number of factors since each factor can be paired with its factor pair.

Thus, if a number has an odd number of factors, then it is a perfect square.

What does this argument prove? Statement P or Q?

To clearly understand this, let us reframe the argument as follows—

- (i) n has an odd number of factors
which implies
- (ii) the existence of a factor-partner pair in which the
same number repeats, say (f, f)
which implies
- (iii) n is a square number: $n = f \times f$

This only proves Q. It doesn't prove that every square number has an odd number of factors (Proposition P).

Is P true?

It is. The argument above can be modified to prove this.

We start with (iii). Clearly, (iii) implies (ii). Does (ii) imply (i)? Not necessarily because (ii) does not state the number of pairs with repeating factors. For example, if the number of such pairs is two—say (f, f) and (g, g) —what can we say about the number of factors? In such a case, the given number will have an even number of factors.

However, a number cannot have more than one factor pair with repeating factors. Thus, a square number necessarily has an odd number of factors. Hence Statement P is also true.

In this example, ' n is a perfect square' and ' n has an odd number of factors' imply each other.

Example 4:

- Proposition P: If two triangles have the same area, then they are congruent.
- Converse Q: If two triangles are congruent, then they have the same area.

Determine if these statements are true or not. Justify the true statements and give a counterexample for each false statement.

These examples show that a proposition can be true but not its converse. Or it can happen that both the proposition and its converse are true.

Think and Reflect

It can also happen that both the proposition and converse are false! Can you give an example?

Example 5: Here is a statement of the Baudhāyana–Pythagoras theorem.

Let a, b, c be the sidelengths of a triangle. If the triangle is right-angled, then $a^2 + b^2 = c^2$.

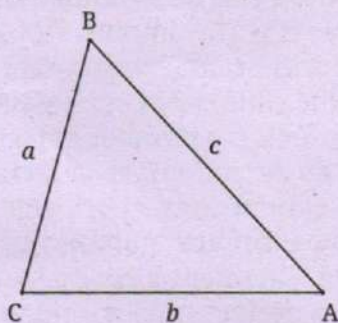
What is its converse?

To write the converse, we keep the first sentence of the Baudhāyana–Pythagoras theorem as it is, and make the necessary change in the second.

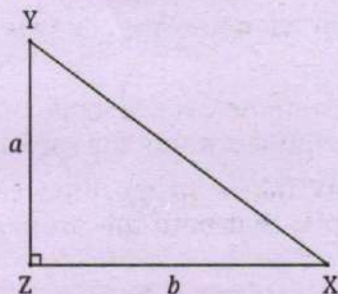
Let a, b, c be the sidelengths of a triangle. If $a^2 + b^2 = c^2$, then the triangle is right-angled.

Is this converse true?

Let $\triangle ABC$ be the triangle with sidelengths a, b, c such that $a^2 + b^2 = c^2$.



Construct another right triangle $\triangle XYZ$ whose perpendicular sides YZ and XZ have sidelengths a and b .



What can we say about the length of XY ?

(Hint: Use the Baudhāyana–Pythagoras Theorem.)

What can we say about $\triangle ABC$ and $\triangle XYZ$?

(Hint: Use the SSS criterion for congruence.)

We conclude that $\triangle ABC$ had to be a right triangle after all.

Hence the converse of the Baudhāyana–Pythagoras Theorem is also true!

A Note

We have seen that 'If X then Y' can also be written as 'X implies Y'. These are not the only ways to express this meaning. It can be conveyed through other words and sentence structures as well. For example, 'Y when X' also gives the same meaning. The following two statements, for instance, have the same meaning.

Proposition: If a number is a perfect square, then it has an odd number of factors.

Proposition: A number has an odd number of factors when it is a perfect square.

Think and Reflect

1. Identify real-life examples in which a proposition is true but not the converse. Give nice counterexamples!
2. Give more examples from geometry as well as number theory in which a proposition is true but not its converse. Give appropriate counterexamples.

EXERCISE SET 9.1

Frame the converse for each of the propositions in Questions 1–12. Then, determine if each of the two statements is true or not. Justify the true statements and give a counterexample for each false statement.

1. If two lines are parallel, then the corresponding angles formed by a transversal are equal.
2. If a quadrilateral is a square, then all its angles are equal.
- *3. Given any $\triangle ABC$, let us bisect the angles at B and C. The bisectors meet at the incentre I of the triangle.

Now extend the bisectors beyond I till they meet the opposite sides at E and F respectively, as shown.

Proposition: If $AB = AC$, then $IE = IF$.

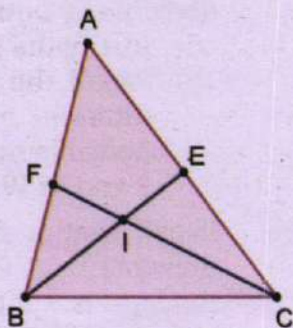


Fig. 9.1

4. If $x = y$, then $a + x = a + y$, where x, y and a are any three numbers. This proposition and its converse are routinely used while solving equations.
5. If a and b are perfect squares, then ab is a perfect square.

In Questions 6 and 7, x and y are real numbers.

6. If $x = y$, then $x^2 = y^2$.
7. If $x = y$, then $x^3 = y^3$.

In Questions 8–12, n is a positive integer.

8. If n is divisible by 24, then it is divisible by both 4 and 6.
9. If n is divisible by 60, then it is divisible by both 5 and 12.
10. If n is the square of a prime number, then it has exactly 3 factors.
11. If n is a product of two unequal prime numbers, then it has exactly 4 divisors.
12. If n and $n + 3$ have no factors in common, then n is not a multiple of 3.
13. There are no known 'neat' expressions that generate only primes! Find counterexamples to the following claims.
 - (i) All numbers of the form $4n^2 + 1$ are prime.
 - (ii) All numbers of the form $n^2 + n + 11$ are prime.
 - (iii) All numbers of the form $4^n + 3$ are prime.
14. Find counterexamples to the following statements.
 - (i) If n is a prime number, then $2^n - 1$ is a prime number.
 - (ii) If n is an even number, then $2^n + 1$ is a prime number.
15. Consider the statement: 'If a number is divisible by 8, then it is divisible by both 2 and 4'.
 - (i) Justify the statement.
 - (ii) Recall the divisibility shortcuts that we have studied for different numbers. To check whether a given number is divisible by 8, is it enough to check whether it is divisible by 2 and 4? Why or why not?
16. Recall that a shortcut to check whether a given number is divisible by 3 is to add the digits of the number and check if the sum is a multiple of 3. Express the relationship between 'a number is divisible by 3' and 'sum of the digits is a multiple of 3' using 'If-then' sentences.

17. We have identified different types of quadrilaterals — squares, rectangles, parallelograms, rhombi, kites and trapezia. One can identify more types (e.g., we could create a category of quadrilaterals that have equal-length opposite sides).

Suppose we have identified a category of quadrilaterals called Q , and we have to construct a quadrilateral of this type. For this, we are to use two thin sticks, put them together as diagonals so that the quadrilateral obtained by joining their endpoints is of type Q (see the Fig. 9.2).

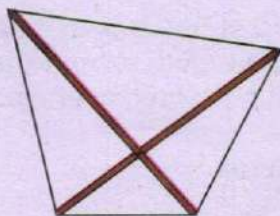


Fig. 9.2

- (i) Suppose Q satisfies the following property.
If a quadrilateral is of type Q , then it has equal-length diagonals.
- (a) Should the two sticks be of equal length? Why or why not?
- (b) Will it matter how the two sticks are put together?
- (ii) Instead of the property mentioned above, suppose Q satisfies the following property.

If a quadrilateral has equal diagonals, then it is of type Q .

What will be your answers to (a) and (b) now?

CHAPTER SUMMARY

- A **proposition** is a statement that is true or false.
- The **converse** of the proposition 'if P then Q ' is the proposition 'if Q then P '.
- It can happen that a proposition is true while its converse is false. It can also happen that both the proposition and its converse are true, or both are false.
- **Counterexamples** are instances that demonstrate the falsity of a given proposition.



10.1 COMBINING THINGS

In earlier grades, we looked at how the average conveys the ‘centre’ of given data. In this chapter, we shall extend the concept of average to a more general setting called ‘weighted average’.

10.1.1 Average of Averages

Example 1: In a badminton academy, there is a group of 11 trainees — 8 seniors and 3 juniors. Their heights and average heights (in cm) are given in the table below. Find the average height of the whole group.

	Heights (in cm)	Average Height (in cm)
Seniors	165, 169, 164, 167, 170, 159, 164, 166	165.5
Juniors	146, 149, 153	149.33

Two students calculate the average height of the whole group in two ways:

Method 1

$$\frac{165.5 + 149.33}{2}$$

$$= \frac{314.83}{2}$$

$$= 157.415$$

Method 2

$$\frac{165 + 169 + 164 + 167 + 170 + 159 + 164 + 166 + 146 + 149 + 153}{11}$$

$$= \frac{1772}{11} = 161.09$$

Whose calculation gives the correct average height of the whole group?

The average height of the whole group is the sum of the heights of all eleven members divided by 11, as in Method 2.

Think and Reflect

Why did Method 1 not work?

Method 1 does not work because it treats the average heights of seniors and of juniors equally—but there are many more seniors than juniors!

Shreyas calculated it differently as

$$\frac{(165.5 \times 8) + (149.33 \times 3)}{8 + 3}$$

$$= \frac{1324 + 448}{11} = \frac{1772}{11}$$

$$= 161.09.$$

Do you understand why this also works?

Can you see why 165.5×8 gives the sum of the heights of all the seniors and 149.33×3 gives the sum of the heights of all the juniors?

This phenomenon is explained below with a generalisation.

Suppose we have two collections of data—Collection 1 and Collection 2.

Suppose the data in Collection 1, having n values, is $x_1, x_2, x_3, \dots, x_n$.

The average of Collection 1 is

$$\frac{x_1 + x_2 + x_3 + \dots + x_n}{n} = a.$$

Thus the sum of the data in Collection 1 is

$$x_1 + x_2 + x_3 + \dots + x_n = an.$$

Suppose the data in Collection 2, having m values, is $y_1, y_2, y_3, \dots, y_m$.

The average of Collection 2 is

$$\frac{y_1 + y_2 + y_3 + \dots + y_m}{m} = b.$$

Thus the sum of the data in Collection 2 is

$$y_1 + y_2 + y_3 + \dots + y_m = bm.$$

The average of the combined data is therefore

$$\begin{aligned} & \frac{(\text{Sum of values in Collection 1}) + (\text{Sum of values in Collection 2})}{(\text{Number of values in Collection 1}) + (\text{Number of values in Collection 2})} \\ &= \frac{an + bm}{n + m}. \end{aligned}$$

Example 2: Jaspreet recently learnt cycling. She has explored different routes in her town. She tracked how much time she cycled on weekdays over the last 3 weeks. Find the mean time spent cycling per weekday over the last 3 weeks.



Well, I did it differently and got the correct answer!



Table 10.1: Time spent cycling, in minutes, by Jaspreet

	Day 1	Day 2	Day 3	Day 4	Day 5	Weekly Average
Week 1	10	13	10	15	16	12.8
Week 2	14	10	20	17	18	15.8
Week 3	15	18	14	20	23	18

Two ways of calculating this are given here.

Method 1

$$\begin{aligned} & \frac{(12.8 + 15.8 + 18)}{3} \\ &= \frac{46.63}{3} \\ &= 15.53 \end{aligned}$$

Method 2

$$\begin{aligned} & \frac{(12.8 \times 5) + (15.8 \times 5) + (18 \times 5)}{5 + 5 + 5} \\ &= \frac{(64 + 79 + 90)}{15} \\ &= 15.53 \end{aligned}$$

The average time spent cycling by Jaspreet on weekdays is 15.53 minutes.

We saw earlier how Method 1 (badminton example) did not produce the correct value. Why does Method 1 give the correct answer in this case? When does Method 1 work and when does it not work? Let us find out.

In general, the average of data in three collections is given as

$$\frac{(\text{Sum of values in Collection 1}) + (\text{Sum of values in Collection 2}) + (\text{Sum of values in Collection 3})}{(\text{No. of values in Collection 1}) + (\text{No. of values in Collection 2}) + (\text{No. of values in Collection 3})}$$

$$= \frac{ap + bq + cr}{p + q + r},$$

where a, b, c are the averages of each collection, and p, q, r are the sizes of the respective collections.

In the cycling scenario, Jaspreet cycled 5 days every week for 3 weeks. This means that all three collections are of the same size, that is 5. Thus, in the general form, the average becomes,

$$\begin{aligned} & \frac{ap + bp + cp}{p + p + p} \text{ (where } p \text{ is the size of the collections)} \\ &= \frac{(a + b + c)p}{3p} = \frac{a + b + c}{3}. \end{aligned}$$

This is the same as adding the 3 weekly averages and dividing the sum by 3. Here, each week has 5 days, so each average comes from the same number of days; that is why treating each week's average equally works. If the weeks had different numbers of cycling days, this method would no longer be correct.

We see that when the sizes of the collections are the same, to find the average of the combined collection, we can add the averages of the collections and divide the sum by the number of collections.

10.1.2 Mixtures

Let us consider what happens when things are mixed.

Example 3: Two glasses of equal quantities of lemonade are prepared. One glass has 10% jaggery and the other has 20% jaggery. If we mix the lemonade from both glasses, what is the concentration of jaggery in the mixture?

Since both the glasses contain equal amounts of lemonade, the mixture will have 15% jaggery. We can also arrive at this mathematically—

Suppose y is the quantity of lemonade in each glass.

The concentration of jaggery in the mixture is

$$\begin{aligned} & \frac{\text{quantity of jaggery in the mixture}}{\text{quantity of the mixture}} \\ &= \frac{\frac{10}{100}y + \frac{20}{100}y}{y + y} \\ &= \frac{0.1y + 0.2y}{y + y} = \frac{0.3y}{2y} = 0.15. \end{aligned}$$

Since the glasses had equal quantities of lemonade, the concentration of jaggery in the combined mixture is midway between the concentrations of jaggery in the individual glasses. This is similar to the cycling example, where all weeks had equal numbers of days.

Example 4: A bowl of 500 mL lemonade has 10% jaggery. A glass of 200 mL lemonade has 20% jaggery. If we mix the lemonade from the bowl and the glass, what is the concentration of jaggery in the mixture?

Can you estimate what part of the mixture is jaggery? Is it 15%, or is it more or less? Why do you think so?

Since the bowl has a larger quantity of lemonade, the concentration of jaggery in the mixture will be closer to 10% (less than midway).

The concentration of jaggery in the mixture is given by

$$\begin{aligned} & \frac{\text{quantity of jaggery in the mixture}}{\text{quantity of the mixture}} \\ &= \frac{500 \times 0.1 + 200 \times 0.2}{500 + 200} \\ &= \frac{50 + 40}{700} = \frac{90}{700} \approx 0.13. \end{aligned}$$

Notice that the final concentration is not the simple average of 10% and 20%. It is closer to 10% because the larger quantity of lemonade comes from the 10% mixture. In general, each concentration influences the final concentration according to the quantity of mixture it contributes.

Example 5: Brass is an alloy of copper and zinc. Batch A of brass weighs 200 kg of which 70% is copper. Batch B weighs 120 kg of which 50% is copper. Batch C is 45% copper. When all three batches are combined, we get an alloy that is 55% copper. What is the weight of Batch C?

The concentration of copper in the mixture is given by

$$\frac{\text{quantity of copper in the mixture}}{\text{quantity of the mixture}}$$

Suppose y is the weight (in kg) of Batch C. From the given information, we know that

$$0.55 = \frac{(\text{Quantity of copper in Batch A} + \text{Quantity of copper in Batch B} + \text{Quantity of copper in Batch C})}{\text{Total weight of brass in all batches}}$$

$$0.55 = \frac{200 \times 0.7 + 120 \times 0.5 + y \times 0.45}{200 + 120 + y}$$

$$0.55(320 + y) = 200 + 0.45y.$$

$$y = 240.$$

Batch C weighs 240 kg.

More generally, if n things with concentrations $x_1, x_2, x_3, \dots, x_n$ and respective volumes/weights $w_1, w_2, w_3, \dots, w_n$ are mixed, the concentration of the mixture is given by

$$X = \frac{w_1 x_1 + w_2 x_2 + w_3 x_3 + \dots + w_n x_n}{w_1 + w_2 + w_3 + \dots + w_n}.$$

Such an average is called the **weighted mean** or **weighted average**. Here, the prefix 'weighted' does not refer to actual weights but abstract weights; however, in some cases, these can be actual weights. That is, the weighted mean is an average where each value is weighted (multiplied) by how important or how big it is. We shall study other contexts where the weighted average is used.

The above formula for the weighted mean was written down by Brahmagupta in his work *Brāhmasphuṭasiddhānta* (c. 628 CE). He used the formula to define the mean depth x of an irregularly-shaped pit. If the pit has depth measurement x_1 for length w_1 , depth measurement x_2 for length w_2 , ... and depth measurement x_n for length w_n , then the mean depth x is given by the above formula.

The concept of weighted arithmetic mean was also used by Indian mathematicians like Śrīdharācārya (c. 750 CE) to estimate the purity of gold-alloys (essentially today's karat calculations).

The concept of the weighted mean eventually appeared in other parts of the world, e.g., Europe, where it gained prominence by the 1700s. The weighted mean now plays a crucial role across mathematics and statistics, allowing us to combine data of different sizes or importance into one number to give an overall accurate picture.

EXERCISE SET 10.1

1. The average score of students on a test in Section A is 72 and that of students in Section B is 76. What is the combined average of both the sections given that Section A has 30 students and Section B has 25 students?
2. A farmer mixes three equal quantities of fertilisers. The first one contains $\frac{1}{10}$ nitrogen, the second contains $\frac{9}{50}$ nitrogen, and the third contains $\frac{3}{60}$ nitrogen. What is the fraction of nitrogen in the mixture?
3. (Śrīdharācārya, Pāṭiṅaṇita, c. 750 CE) In ancient India, Varṇa was the measure of gold purity. A purity of 16 varṇa meant pure gold; in general, a purity of k varṇa meant that the gold-alloy was $k/16$ gold and the rest impurities. (Now the term used is karat; 16 Varṇa = 24 karat.)

Suppose a goldsmith melts together three pieces of gold: 9 units at 12 varṇa, 5 units at 10 varṇa, and 17 units at 11 varṇa. Find the purity in varṇa of the combined gold.

4. The average rainfall per day in the months of May, June, and July in a certain location are 3.5 mm, 10 mm and 8.7 mm respectively. Write an expression that gives their combined average.
5. Calculate the concentration of spice mix in these two scenarios.
 - (i) A 100 mL *kashayam/kadha* with 5% spice mix, a 200 mL one with 10% spice mix, and a 300 mL one with 15% spice mix are combined.
 - (ii) A 300 mL *kashayam/kadha* with 5% spice mix, a 200 mL one with 10% spice mix, and a 100 mL one with 15% spice mix are mixed.

10.1.3 Custom Weights

The weighted mean is also used whenever parts of data require different levels of importance or weightage. Let us look at an example.

Example 6: Rehmat's marks in Maths are as follows: 60% in internal tests, 64% in the project, 73% in the final exam. The annual percentage score is calculated by combining the internals, project, and final exam in the ratio 3 : 2 : 5. What is Rehmat's annual score in Maths?

Suppose all modes of assessment had an equal weightage. Then the annual score can be obtained by calculating the arithmetic mean of 60%, 64%, and 73%, which is $\frac{60 + 64 + 73}{3} = 65.6\%$.

The given ratio 3:2:5 indicates the relative weightage and importance that each form of evaluation gets. The data with these relative weights is equivalent to 60, 60, 60, 64, 64, 73, 73, 73, 73, 73. The arithmetic mean is given by $\frac{60 \times 3 + 64 \times 2 + 73 \times 5}{10} = 67.3\%$.

This can also be expressed as a weighted mean of 60%, 64%, 73% with weights 3, 2, 5 respectively:

$$\frac{\left(\frac{60}{100}\right) \times 3 + \left(\frac{64}{100}\right) \times 2 + \left(\frac{73}{100}\right) \times 5}{3 + 2 + 5} = \frac{\left(\frac{673}{100}\right)}{10} = 67.3\%.$$

The weights assigned to the values in any data indicate the relative importance/emphasis that each value gets with respect to the others.

EXERCISE SET 10.2

- Savitri's marks in Kashmiri are as follows: 35 out of 50 in internal tests, 44 out of 60 in the project, and 80 out of 100 in the final exam. The annual percentage score is calculated by combining the internals, project, and final exam in the ratio 3 : 4 : 5. Which of the following expression(s) gives her annual score (as a percentage) in Kashmiri?

(i) $\frac{35 \times 3 + 44 \times 4 + 80 \times 5}{3 + 4 + 5}$

(ii) $\frac{\left(\frac{35}{100}\right) \times 3 + \left(\frac{44}{100}\right) \times 4 + \left(\frac{80}{100}\right) \times 5}{3 + 4 + 5}$

(iii) $\frac{\left(\frac{35}{50}\right) \times 3 + \left(\frac{44}{60}\right) \times 4 + \left(\frac{80}{100}\right) \times 5}{3 + 4 + 5} \times 100$

$$(iv) \frac{\left(\frac{35}{50} \times 100\right) \times 3 + \left(\frac{44}{60} \times 100\right) \times 4 + \left(\frac{80}{100} \times 100\right) \times 5}{3 + 4 + 5}$$

Here are a few more situations where the concept of weighted mean is relevant:

1. **Ratings of a hotel/eatery:** The experience of eating at a restaurant is influenced not only by the taste or quality of the food but also by the ambience and the service provided. Every customer is asked to rate each of these aspects. The customer's combined rating is determined by the relative weights assigned to food, service, and ambience. Suppose one of the customers has given a rating as shown in the picture. Suppose the weights assigned to food, service, and ambience are 4:3:2 respectively. The user's rating for the restaurant is $\frac{5 \times 4 + 3 \times 3 + 4 \times 2}{4 + 3 + 2} = 4.11$.

MEAN HOTEL	
Food	★★★★★
Service	★★★★☆
Ambience	★★★★☆

2. **Ratings of a product:** A store makes and sells portable furniture. The store owners make changes to their products regularly. To calculate the store's average rating, it makes sense for them to give their current products more weight than older products. For example, they may choose to assign twice the weightage to customer reviews in the last 3 months compared to older reviews.

We use weighted mean whenever we want the combined result to reflect how much each part contributes to the whole—by amount, time, or importance—rather than just counting each part equally.

EXERCISE SET 10.3

1. A stationery shop owner made ₹8000 selling books, of which 30% is the profit amount, and ₹1000 selling book covers, of which 50% is the profit amount. What is the percentage of profit on the total sales?
2. A white stork's migration is tracked. The average daily distance travelled, calculated over 20 days, is 44.5 km. On the 21st day, it flew 55 km. What is the average daily distance travelled over these 21 days? Make a guess before you calculate.

3. A 600 mL solution with 5% salt is mixed with a 300 mL solution with 8% sugar. What are the concentrations of salt and sugar in the mixture?

- (i) Salt: 5%, Sugar: 8% (ii) Salt: 13%, Sugar: 3%
 (iii) Salt: 6%, Sugar: 6% (iv) Salt: 5.55%, Sugar: 8.88%
 (v) Salt: 3.33%, Sugar: 2.67% (vi) Salt: 4.1%, Sugar: 7.08%

4. At a *panipuri* (*golgappa*) stall, the concentration of spice in the *pani* (spiced water) was 8%. Many customers complained that it was too spicy. What quantity of regular water should be mixed into the 10 litres of *pani* so that the spice level is reduced to $\left(\frac{3}{4}\right)^{\text{th}}$ of the original concentration?



5. A physical fitness evaluation is being undertaken. The final marks are calculated by combining the marks for strength, flexibility, and agility in the ratio 4:5:6. Rashi has scored 60, 65, and 70. Keerthi has scored 55, 65, and 75 respectively.
 (i) Find out whose total is more without calculating.
 (ii) What are their final marks?
6. A restaurant collected ratings from 10 customers on a scale of 1 to 5. The resulting data is shown in the table below.

	5★	4★	3★	2★	1★
Food	5	3	2	0	0
Ambience	0	4	5	1	0
Service	1	2	2	4	1

What is the average rating if the metrics are combined with the weights food : ambience : service = 6 : 5 : 4?

7. The following table shows the weight data of langurs in an animal facility. Without doing any computations, can you tell

whether there are more male langurs or more female langurs?
Or are they equal in number?

	Male langurs	Female langurs	All langurs
Average weight	16.5 kg	13.8 kg	14.925 kg
Number of langurs			60

- (i) Which of the following expression(s) describes the given scenario?

(a) $\frac{16.5x + 13.8y}{x + y} = 14.925$

(b) $\frac{16.5x + 13.8y}{60} = -14.925$

(c) $\frac{16.5x + 13.8y}{2} = -14.925$

(d) $\frac{16.5x + 13.8y}{16.5 + 13.8} = -14.925$

- (ii) Find out how many male langurs are present.
 (iii) A female langur weighing 15.2 kg is admitted to the facility. What is the average weight of the female langurs after this?
 (iv) Two male langurs weighing 16.9 kg and 16.1 kg are released from the facility. What is the average weight of the male langurs after this?
 (v) Now, suppose one of the male langurs lost 1 kg of weight. What is the average weight of all the male langurs after this?

8. Dorjee has collected 1 litre of water from the Dead Sea! Using the information in the table, answer the following questions. A calculator can be used if necessary.

Water Source	Salinity
Dead Sea	$\approx 34\%$
Other seas and oceans	$\approx 3.5\%$
Ground water	$\approx 0.01\%$
Purified drinking water	$\approx 0.001\%$



- What is the salinity of the mixture if he mixes 1 litre of water from the Dead Sea with 2 litres of purified drinking water?
- Is it possible to mix water from the Dead Sea and purified drinking water to get a mixture with the salinity of groundwater? Why/Why not? What quantity of purified drinking water should Dorjee mix with 1 litre of water from the Dead Sea to get a mixture having the salinity of groundwater?
- Is it possible to mix water from the Dead Sea and groundwater to get a mixture with the salinity of purified drinking water? Why/Why not? What quantity of groundwater should he mix with 1 litre of water from the Dead Sea to get a mixture having the salinity of purified drinking water?

10.2 VISUALISING AND INTERPRETING DATA

10.2.1 Stacking Columns

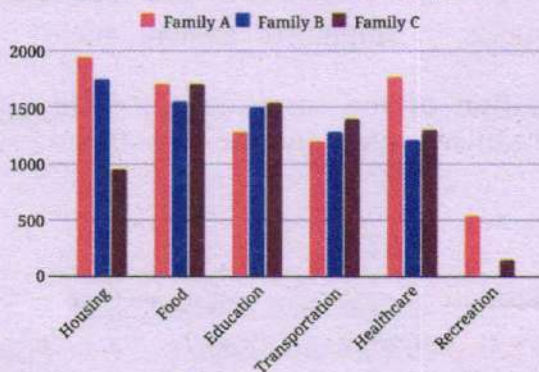
The following table shows the average weekly expenditure, in rupees (₹), of three families across categories. What would a cluster-column chart for this data look like?

Table 10.2

	Housing	Food	Education	Transportation	Healthcare	Recreation
Family A	1940	1700	1280	1200	1770	535
Family B	1750	1546	1500	1280	1210	0
Family C	950	1700	1540	1400	1300	150

There are two possibilities depending on the choice of the cluster; the clusters can be categories or families:

Comparing expenditure of families across categories



Comparing category-wise expenditure of each family

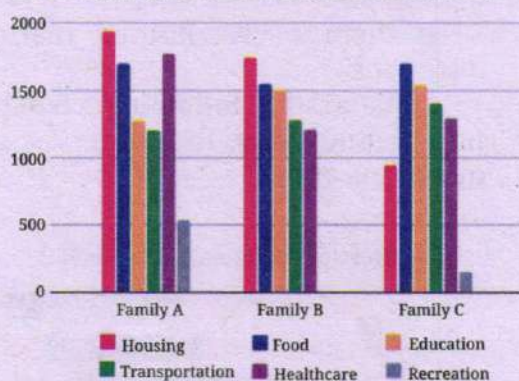


Fig. 10.1: Two cluster-column graphs showing different ways of visualising the data in Table 10.2

Answer the following questions based on the charts above. Try to answer each question by looking at Choice 1 first, and then Choice 2. Which was easier?

1. Which family spent the least on housing?
2. Which family spent the most on healthcare?
3. Approximately how much did Family A spend on education?
4. Which category did Family B spend most on?
5. Did Family A spend more on food or on healthcare?

You may have found Choice 1 more suitable to answer Questions 1 and 2, and Choice 2 more suitable to answer Questions 4 and 5.

The choice of the cluster-column chart depends on what the primary focus is — comparing different families' expenditures in each category (Choice 1) or comparing category-wise expenditures within each family (Choice 2).

Find the answers to the following questions.

6. Which family spent the least amount overall?
7. Approximately how much did Family C spend overall?

8. Approximately what percentage of Family C's expenses were on housing?

Was it as straightforward to answer these questions as the earlier questions?

Is there a visualisation that makes it simpler to answer such questions?

Look at the following visualisation of the same data. Try to understand how the chart is organised and answer the three questions above.

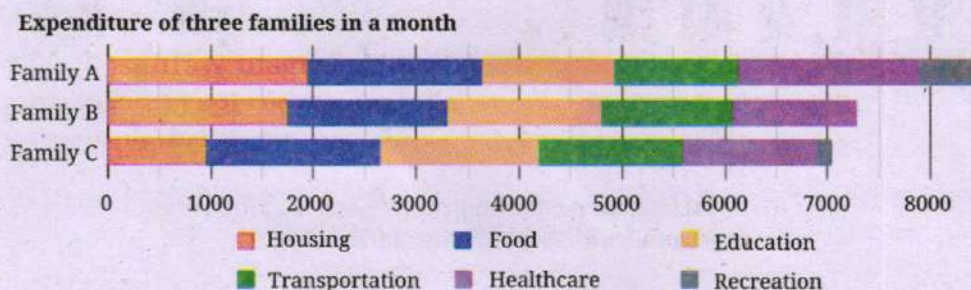


Fig. 10.2: A Stacked Bar Chart of the data in Table 10.2

Such a graph/chart is called a **stacked bar or column chart**. Bars within a cluster are joined one after the other in a particular order. **Stacked bar charts are used when we want to compare the totals but also want to keep track of the components that make up the totals.** Here, the total length of a full stacked (combined) bar represents the total expenditure of a family in that month. Also, the length of a smaller bar and the full stacked bar can be compared to get an estimate of what fraction or percentage the smaller bar makes up of the full stacked bar.

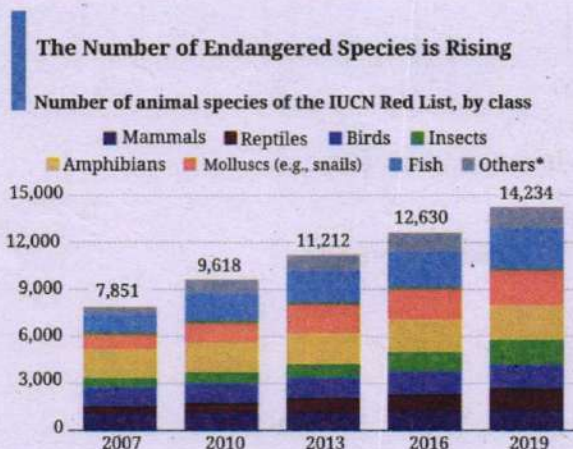
Now, use the stacked bar chart to answer the Questions 1–5 shown earlier.

Is it possible? Is it easy? If not, what made it harder than before?

Except for the first small bar, the rest of the stacked small bars do not begin from 0. This makes it a little harder to estimate the quantity corresponding to a bar. Also, bars across clusters need not have a common base or a starting point. So, comparing bars within clusters and across clusters may be harder. However, seeing the totals of all the clusters is easier.

EXERCISE SET 10.4

1. The following stacked column chart shows the number of animal species in the IUCN Red List, by class, over time.



* Other invertebrate (spineless) animals, such as crustaceans, corals and arachnids (spiders, scorpions)

Source: IUCN Red List



statista

The International Union for Conservation of Nature (IUCN) Red List is a large, regularly-updated database that tells us the extinction risk of species across the world. It classifies species into groups such as Least Concern, Vulnerable, Endangered, Critically Endangered, Extinct in the Wild and Extinct, based on scientific data about their numbers, trends and habitats.

- What does the number 14,234 on the top of the column 2019 represent?
- Approximately how many reptile species were in the list in 2016?
- Which class(es) of species have seen a relatively small increase in count between 2007 and 2019?

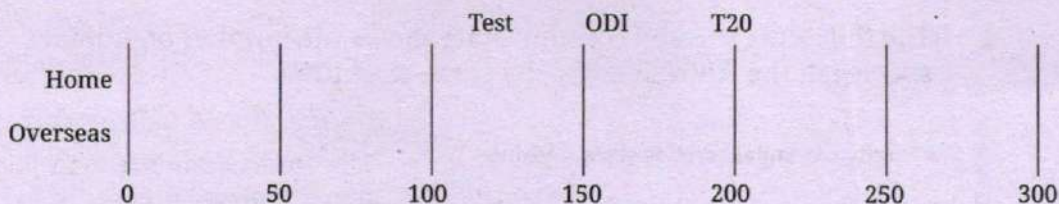
2.

	Test	ODI	T20
Home	62	100	32
Overseas	172	49	71

The wickets taken by a bowler in International Cricket matches till 2025 are shown in the table. Complete the given stacked bar charts (the bar lengths can be approximate),

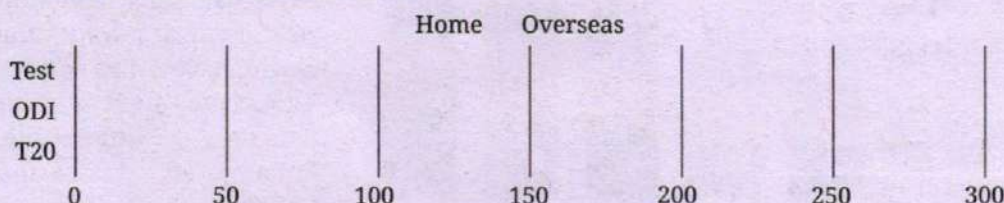
(i) comparing the total wickets taken at home vs. overseas:

Wickets taken at home vs. overseas



(ii) The total wickets taken in each format:

Wickets taken in different formats



10.2.2 An Alternative to Pie Chart

Try answering the following question based on the expenditure stacked bar chart you saw earlier:

Which family has the smallest share of their total expenditure towards healthcare?

Was it easy?

If not, what visualisation can help answer such questions easily?

Last year, we learnt how pie charts can capture proportions of quantities. Let us visualise this data through pie charts:

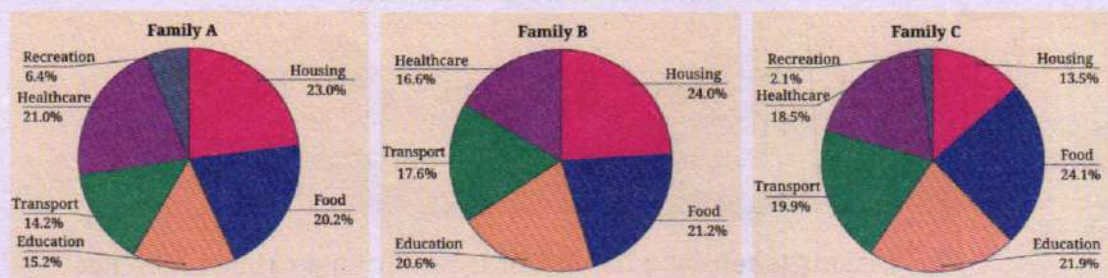


Fig. 10.3: Pie charts of data of each family in Table 10.2

Now, consider the following visualisation showing proportions for the same data:

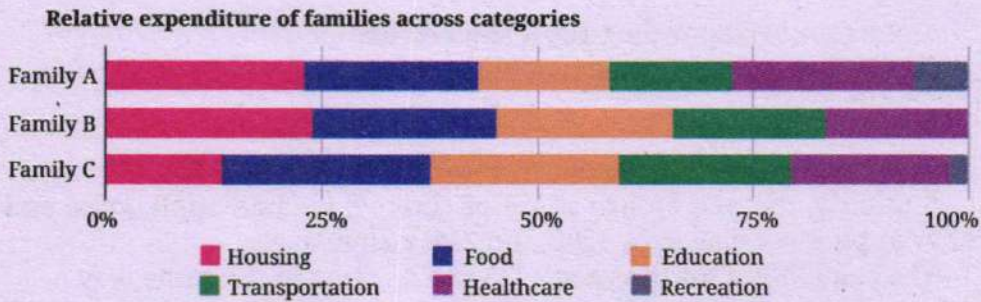


Fig. 10.4: A 100% stacked bar chart of data in Table 10.2

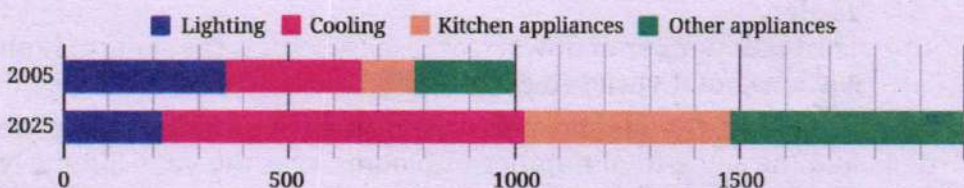
Here, the total expenditure of each of the three families is shown by a stacked bar of a fixed length. The lengths of the smaller bars in each stacked bar are proportional to the corresponding expenditures of that family. Such a chart is called a **100% stacked bar chart**. It is similar to a pie chart—the complete angle in a pie chart is split proportionately based on the quantities, whereas here, a fixed length is split proportionately.

In pie charts, comparing angles in different orientations may be difficult. It may be easier to compare lengths across stacked bars. However, even in 100% stacked bar charts, if the lengths of a particular bar are similar across the stacked bars but the location is different, comparison may be difficult, as in the chart above.

Powering Up!

Example 7: The average category-wise daily electricity consumption of a house, in kWh, in 2005 and 2025 is shown in the table below. The corresponding stacked bar chart is also shown.

	Lighting	Cooling	Kitchen appliances	Other appliances
2005	360	300	120	220
2025	220	800	460	520



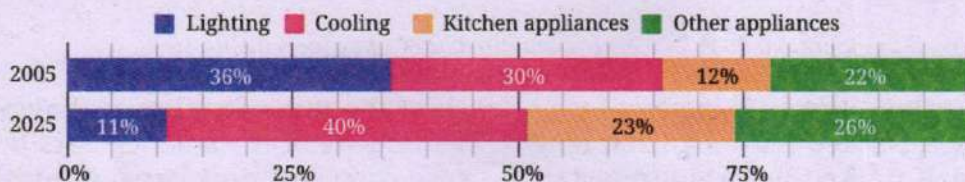
Make the corresponding 100% stacked bar chart.

In 2005, the total unit consumption was $360 + 300 + 120 + 220 = 1000$ units.

The percentage share of lighting in this was $\frac{360}{1000} \times 100 = 36\%$.

Similarly, the percentage share of cooling, kitchen appliances, and other appliances was 30%, 12%, and 22% respectively.

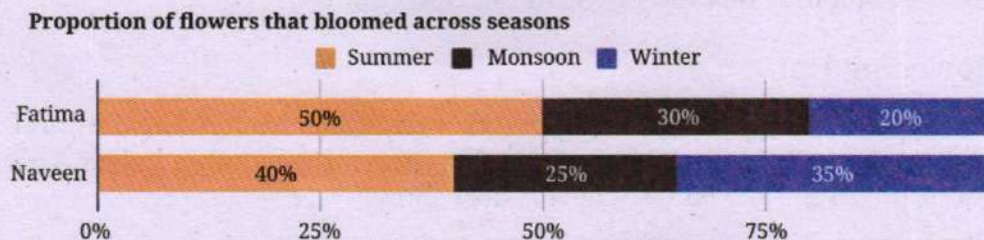
The calculation for the year 2025 can be done in the same way.



What do you observe? What do you find interesting in the data/graphs? Discuss with others.

What Can We Infer?

Example 8: The 100% stacked bar below compares the proportion of flowers blooming in Fatima's garden and Naveen's garden across seasons.



Which of the following statements can be inferred from this chart?

- In Fatima's garden, there were more blooms in the summer than in the monsoon.
- In summer, Fatima's garden had more blooms than Naveen's garden.
- In winter, Fatima's garden had fewer blooms than Naveen's garden.
- The total number of flowers, across seasons, is the same in both gardens but it varies in each season.

(Hint: Find out the blooms per season in each garden if both the gardens had the same total number of blooms over the year. Similarly find out what happens if the totals are very different.)

What does the 100% stacked bar tell us?

In each garden, the total number of flowers for the whole year is shown as one bar and is taken as 100%. The colours show what fraction of that total bloomed in each season. So for Fatima's garden, the bar tells us: 50% of her flowers bloomed in summer, 30% in monsoon, and 20% in winter. For Naveen's garden, the bar tells us: 40% in summer, 35% in monsoon, and 25% in winter.

Statement (i): It compares seasons within the same garden (Fatima's). So, within Fatima's garden, the summer part is the largest fraction. This means she actually had more flowers in summer than in monsoon. So, (i) must be true.

Statements (ii) and (iii): These compare across gardens in the same season.

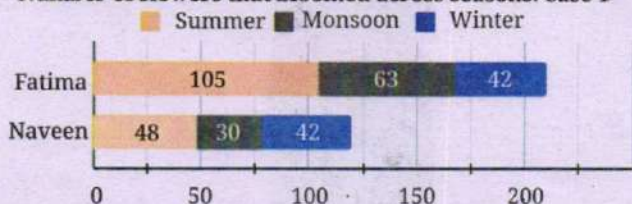
For example, in statement (ii), Fatima has a higher percentage in summer (50%) than Naveen (40%). But the chart does not tell us how many flowers they each had in total. Naveen's total could be much larger. Then even 40% of Naveen's total might still be more flowers than 50% of Fatima's total. That means we cannot be sure who has more flowers in that season. Hence, (ii) and (iii) cannot be inferred from this graph.

Statement (iv): A 100% stacked bar chart always shows each bar as '100%' of that garden's total. It does not tell us whether the totals for the two gardens are equal or not. Many different total numbers can give the same 100% bar. Therefore, we also cannot say anything about the total over the whole year from this graph.

A 100% stacked bar chart helps us compare fractions/percentages inside each garden, but it does not tell us the actual numbers of flowers.

Two possible stacked bar charts are shown below for the 100% stacked bar chart we just saw.

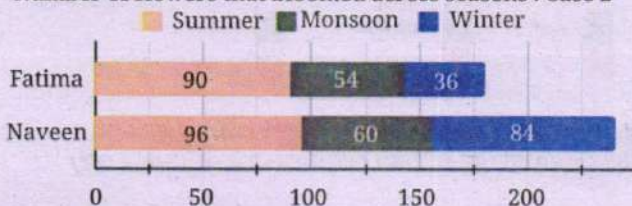
Number of flowers that bloomed across seasons: Case 1



$$\frac{105}{210} \times 100 = 50\% \quad \frac{63}{210} \times 100 = 30\% \quad \frac{42}{210} \times 100 = 20\%$$

$$\frac{48}{120} \times 100 = 40\% \quad \frac{30}{120} \times 100 = 25\% \quad \frac{42}{120} \times 100 = 35\%$$

Number of flowers that bloomed across seasons : Case 2



$$\frac{90}{180} \times 100 = 50\% \quad \frac{54}{180} \times 100 = 30\% \quad \frac{36}{180} \times 100 = 20\%$$

$$\frac{96}{240} \times 100 = 40\% \quad \frac{60}{240} \times 100 = 25\% \quad \frac{84}{240} \times 100 = 35\%$$

Notice that the percentage of flowers that bloomed in summer is more in Fatima's garden but this doesn't say anything about the actual number — Fatima's garden has more blooms than Naveen's in summer in Case 1 but her garden has fewer blooms than his in Case 2. Also, in Case 1 both Fatima's and Naveen's garden had 42 blooms but the respective percentages are different.

Think and Reflect

1. What does this say about the scope of the stacked bars and 100% stacked bars?
2. Given a stacked bar chart can we make a corresponding 100% stacked bar chart?
3. Given a 100% stacked bar chart can we make a corresponding stacked bar chart?
4. What kind of inferences or comparisons can be made from a stacked bar chart and in a 100% stacked bar chart?

A Typical Day

Example 9: The following chart shows the time spent on different activities in a day by the average Indian. What do you notice? What do you wonder about?

You may point out that you don't do much work and that your parents don't spend any time learning. Note that this strip is representative of people across ages, genders, regions, social/family backgrounds, etc., across India.

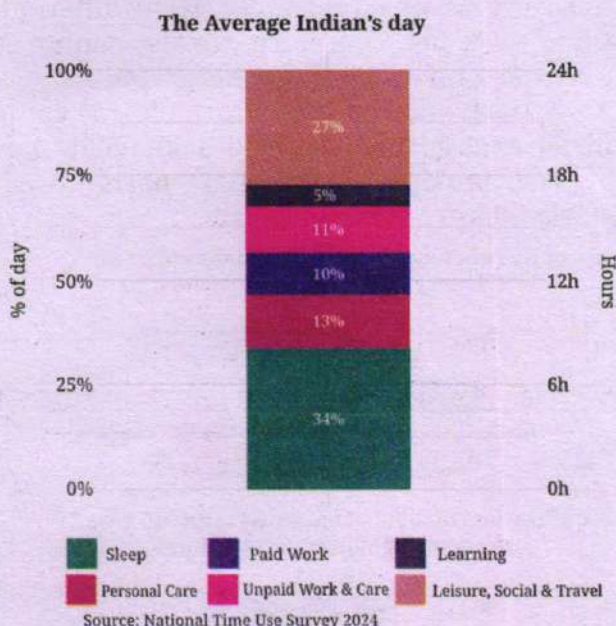


Fig. 10.5: Time spent on different activities in a day by an average Indian (Values are rounded off)

When we visualise the same data by grouping ages, we get the following stacked column chart:

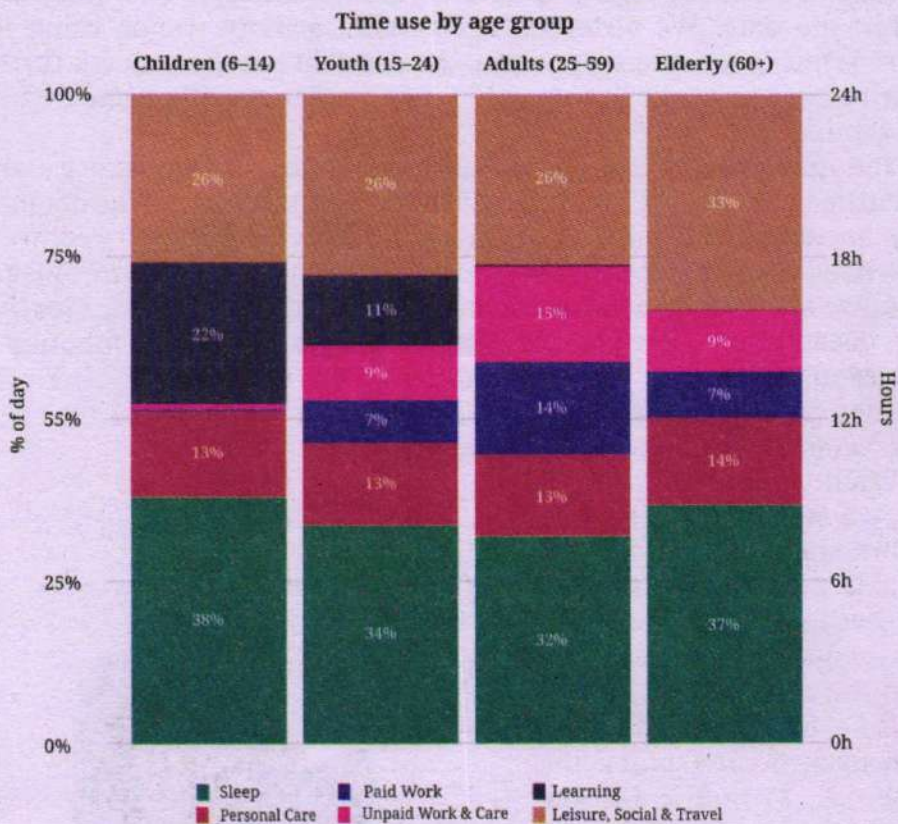


Fig. 10.6: Age-wise grouping of data (Values are rounded off)

Think and Reflect

1. What do you find interesting in the chart above? What can you infer? Discuss.
2. Do you remember the sleep time over age trend that you studied last year? Does that trend align with this chart?
3. Can you explain why the learning time of people in the age group 15 – 24 has reduced significantly compared to that of the age group 6 – 14?
4. Do adults spend about an equal amount of time in paid and unpaid work? What do you think?

There might be a very small fraction of people whose day is similar/close to that of the average Indian. As we saw here, looking at only the average of a diverse collection often hides the differences and nuances within the data. We observed age-specific activity trends using this chart, which gave us a better idea. We could break these down further with respect to gender to examine any gender-specific trends within age groups.

You may wonder if there are differences across regions (e.g., rural and urban), or income, etc. You may be curious to look into the details—such as, what is included under personal care for different groups, or how does time spent on travel vary across cities and towns/villages? What we spend time on also depends on the times we live in. Clearly, a few decades ago, people didn't have access to mobile phones or televisions. Only time will tell what such charts will look like after a few decades. We may also ask—how do people in other countries spend their time? How similar or different are the trends?

What type of chart is this—a stacked bar chart or a 100% stacked bar chart?

It is both! It is a stacked bar because each small bar represents the amount of time spent on a certain activity. It is also a 100% stacked bar as each stacked bar's total is the same, i.e., 24 hours, and the lengths of the smaller bars within and across the stacks also represent the relative proportions of time spent.



If there is only one stacked bar, it can also be read as a 100% stacked bar!!

Think and Reflect

Do you remember the 'What Can A Strip Say?' activity from last year? In each strip, if we club the tiny strips belonging to each activity together, will we get a 100% stacked bar chart like the one in Fig. 10.6?

EXERCISE SET 10.5

1. The smart watch data of 5 people was tracked from Monday to Friday. The recorded data tells us how many hours each person spent lying down, sitting, and standing or moving around, etc., as shown in the following table.

Table 10.3: Average number of hours spent per day (Monday–Friday) lying down, sitting, and standing or moving around, etc., of 5 people

	Lying down	Sitting	Standing or Moving around etc.
Pavani (Patient)	20	3	1
Raghu (Nurse)	7	4	13
Zakir (Teacher)	8	6	10
Sahana (Student)			
Julie ()	8	10	6

- How much time does Sahana spend per day in each of these body-states? Make a reasonable guess and fill her row in the table.
- Guess what activity or work Julie could be engaged in based on the data. Find out what your classmates' guesses are.
- Complete the chart below based on the tabular data.



Think and Reflect

1. Will the bars look similar if a different teacher's data is considered instead of Zakir's?
2. Will the bars look different if the data for all 7 days of the week is considered?
3. Some bars in the chart may also match people doing other kinds of work/activities. Can you think of any? Discuss.

END-OF-CHAPTER EXERCISES

1. In cricket, the **run rate** is the average number of runs scored per over. In a T20 match, a team scored 6 runs in the first over making the run rate 6.
 - (i) In the second over they scored 12 runs. What is the run rate now?
 - (ii) After Over 19, their run rate was 6. What is the run rate after 20 overs given the team made 12 runs in the last over?
2. Five friends who play badminton surveyed the city and collected the price of shuttlecocks across brands and types (Nylon and Feather). The following is the list of the prices, in rupees (₹), that each one compiled.



Nylon shuttle
(N)



Feather shuttle
(F)

Yusuf: 40(N), 105(N), 383(F), 108(N), 165(F), 116(F)

Srikanth: 194(N), 85(N), 93(N), 121(N)

Kashvi: 49(N), 297(F), 105(N), 275(F), 40(N)

Prasanna: 124(N), 333(F), 182(N), 258(F)

Gracy: 220(F), 458(F), 129(F), 183(N)

Can you describe a way they can work together to find the average price of a shuttlecock across types?

- (i) Each one calculated the average of the prices they had gathered. Suppose these are a_v , a_s , a_k , a_p , a_g respectively. Write an expression that gives the combined average.
- (ii) Suppose a_n , a_f are the average prices of the nylon shuttlecocks and the feather shuttlecocks, respectively. Write an expression that gives the average price of a

shuttlecock. Will this be equal to the answer we got using the method in the previous part?

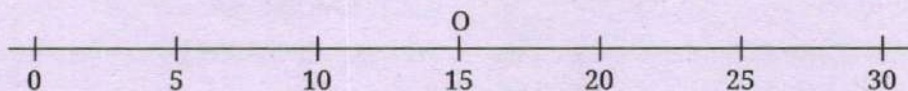
3. Shreyas holds 25 shares of a company at an average price of ₹150 and Vaishnavi holds 5 shares of the same company at an average price of ₹150.

- Shreyas buys 10 shares of this company at a price of ₹30 each. What is the average price per share for him after the purchase?
- Vaishnavi buys some shares of this company at a price of ₹30 each and the average price per share for her after the purchase is ₹70. How many shares did she buy?

4. (Prthūdakasvāmī, commentary on Brahmagupta's Brāhmasphuṭasiddhānta, c. 864 CE) A "hasta" (meaning "forearm") refers to a unit of length measuring around 18 inches.

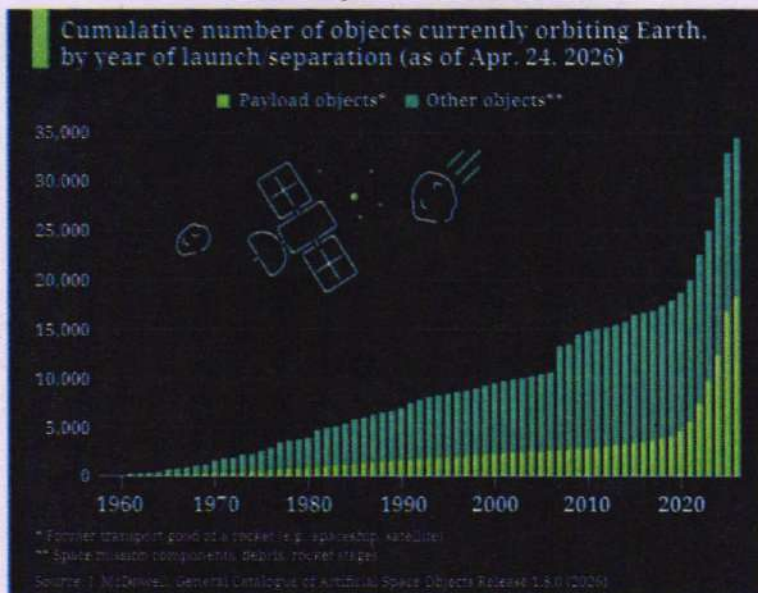
A pool 30 hastas long is dug to different depths along its length. It is divided into 5 sections having lengths 4, 5, 6, 7, and 8 hastas, and is dug to depths of 9, 7, 7, 3, and 2 hastas, respectively. Find the mean depth of the pool.

5. Suvarna had purchased 1g gold at ₹15k (short for ₹15,000). This is shown by point O denoting the average price of gold that she possesses. Six different scenarios are given below for her next transaction. For each scenario estimate and mark the average price of gold she will have after the transaction.



- Purchase 1g gold at ₹30k
 - Purchase 2g gold at ₹30k
 - Purchase 1g gold at ₹10k
 - Purchase 10g gold at ₹10k
 - Purchase 0.5g gold at ₹30k
 - Purchase 0.5g gold at ₹15k
6. Given some data with corresponding weights, how would the weighted average change if all the weights are doubled? If required, experiment with some data. What do you observe? Justify your answer using algebra.
7. Answer the following questions based on the graph.
- In 2003, approximately how many total objects were found orbiting Earth in space? In what year did this number double?
 - Find the approximate number and share of payload objects and other objects in the year 2025.
 - What can you say about the number of payload objects over time, and the number of other objects over time? What can

you say about the share of payload objects over time, and the share of other objects over time?

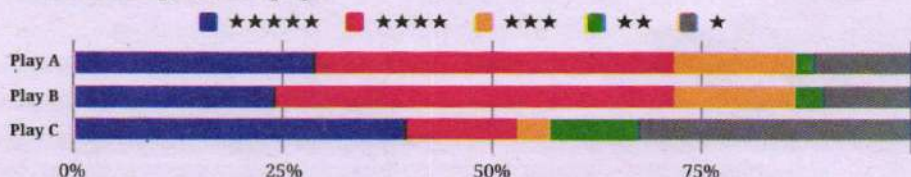


- (i) Identify the correct inference(s) from the statements given.
 - (a) More schools have a playground in Haryana compared to Uttarakhand.
 - (b) Approximately every 2 out of 3 schools in Arunachal Pradesh have a playground.
 - (c) Punjab has the highest number of schools with a playground.
 - (d) Suppose it is given that Maharashtra has more schools than Telangana. Then the number of schools having a playground is more in Maharashtra.
- (ii) Using the information given, can we find the nation-wide percentage of schools with a playground? If not, what additional information is needed?

9. Decision Dilemma:

- (i) Which of the plays would you choose to watch based on the following chart?

Share of ratings of three plays



- (ii) The corresponding stacked bar chart is shown below. Would you change your decision after looking at this chart? Why/Why not? Discuss.

Ratings of three plays



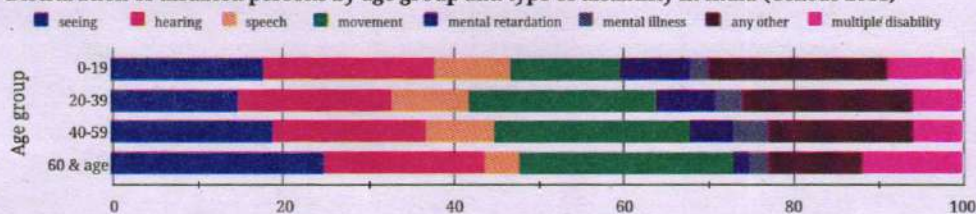
10. A triathlon is an endurance race consisting of swimming 3.8 km, cycling 180 km, and running 42.2 km. Athletes compete for the fastest overall time, completing each segment in that order. The following table shows the finish time of 3 athletes in each segment.

Table 10.4: Finish time of three athletes in each segment of a triathlon race (values in hh:mm)

	Swimming	Cycling	Running
Athlete 1	01:08	05:00	03:15
Athlete 2	01:05	05:10	03:35
Athlete 3	01:22	05:55	03:50

- What is a suitable representation of this data — a stacked bar chart or a 100% stacked bar chart? Why do you think so?
 - Suppose a 100% stacked bar chart is drawn. Which of the following question(s) can be answered by looking at just that chart?
 - Who finished the race first?
 - Approximately what fraction of their race time did Athlete 1 spend cycling?
 - Which athlete took the longest for running?
11. Look at the following graph. What do you notice? What do you wonder? Write your inferences.

Distribution of disabled persons by age group and type of disability in India (Census 2011)



12. **Individual project:** Do at least one of the following.
- Recall the previous day and fill in the approximate time spent lying down, sitting, and standing/moving around, etc. Ask at least 2 of your family members about their day and fill it in. Alternatively, you can choose to track just your own body-state for any one of the weekdays, Saturday, and Sunday. Visualise it using a stacked bar chart. Discuss with your class how you arrived at the estimated time spent in each state.
 - Visualise your family's monthly expenditure using a 100% stacked bar chart.
 - First, identify the expense categories in your family budget.

- (b) Discuss with your family members to decide how you will collect and organise the data.
 - (c) Collect expense data for at least 3 months.
 - (d) Represent the data in a 100% stacked bar chart, with each bar showing how the total monthly expenditure is divided across the different categories.
 - (e) Finally, write your observations and inferences based on the chart.
13. **Small-group project:** Make a group of 3–4 students. Choose one scenario to design a custom rating scheme by assigning appropriate weights: (a) shopping experience at a cloth store, (b) travel experience in a bus, (c) tourism experience of a nearby tourist spot, (d) clinic/hospital experience
- (i) Discuss and arrive at 4–6 aspects to rate. For each aspect chosen, justify why it matters for an overall experience.
 - (ii) Discuss and decide the relative weights and justify.
 - (iii) Collect or imagine ratings from at least 10 people on a scale of 1–5 (5 being very good).
 - (iv) Compute each individual rating. Compute the overall average across individuals.
 - (v) Make a 100% stacked bar chart using your data.
 - (vi) Write a short note on the observations and inferences, what your rating system captures well and what it misses.
14. **Whole class project:** Each student shares the average age of their family and the number of family members. Discuss among the class and come up with a way to find out the average age of all the families of the class.
- *15. Given some data with corresponding weights, what would happen to the weighted average if all the weights are increased by a constant value, say 1? If required, experiment with some data. What do you observe? Justify your answer using algebra.
- *16. A farm has some cows, sheep, and chickens. Last year the cows made up 60%, the sheep 25%, and the chickens 15%. There was a decrease in the number of all three animals' population over the year. Choose the possibilities for the change in their respective shares of the population —

- (i) % of cows decreased, % of sheep decreased, % of chickens decreased
- (ii) % of cows increased, % of sheep increased, % of chickens increased
- (iii) % of cows remained the same, % of sheep remained the same, % of chickens remained the same
- (iv) % of cows decreased, % of sheep increased, % of chickens remained the same
- (v) % of cows increased, % of sheep increased, % of chickens decreased.

CHAPTER SUMMARY

- The **weighted average** or **weighted mean** of $x_1, x_2, x_3, \dots, x_n$ with weights $w_1, w_2, w_3, \dots, w_n$ respectively is given by

$$X = \frac{w_1 x_1 + w_2 x_2 + w_3 x_3 + \dots + w_n x_n}{w_1 + w_2 + w_3 + \dots + w_n}.$$

- An average of averages can be computed using the weighted mean. The weighted mean can also be used to calculate the mean when certain values are added to or removed from data.
- The weighted average is also useful in calculating the concentration of mixtures.
- Whenever different parts of data have different number of values or levels of importance, the weighted mean can be used by assigning appropriate weights to each part of the data. Examples include rating systems and combining marks of different assessments to get a final score.
- **Stacked bar charts** may be used in place of cluster bar charts if the total of each cluster is also a matter of interest.
- **100% stacked bar charts**, like pie charts, compare proportions and not absolute values.
- Given a stacked bar chart, a corresponding 100% stacked bar chart can be made but not vice versa.



These days the word ‘algorithm’ pops up everywhere. What exactly is an algorithm? What does it have to do with mathematics?

11.1 ADDING NUMBERS

We first learned to add numbers by counting the sum explicitly. To add 5 and 7, we would draw 5 dots to represent the number 5, then draw 7 more dots to represent the number 7, and finally count the dots, 12, to get the answer. Imagine adding large numbers this way, for instance $473 + 695$!



$$\begin{array}{r} 473 \\ + 695 \\ \hline 8 \end{array} \quad \begin{array}{r} 1 \\ 473 \\ + 695 \\ \hline 68 \end{array} \quad \begin{array}{r} 1 \\ 473 \\ + 695 \\ \hline 1168 \end{array}$$

Fortunately, we discovered a much faster way to add numbers. We wrote the numbers one below the other in the Indian number system, with the rightmost digit aligned. We then added each column from right to left, remembering to add a carry to the left if the total was equal to or more than 10.

11.1.1 Adding Numbers Digit by Digit

This is an example of an **algorithm**, a step-by-step procedure to arrive at an answer. We describe an algorithm by writing down the steps as precisely as possible. For instance, here is one way to describe our addition algorithm.

Algorithm to add two numbers

1. Write the numbers one below the other in the Indian base-ten place-value system so that the digits are aligned from right to left.
2. Add the rightmost digits.
 - If the sum is less than 10, write the sum directly below the two digits. Set the value of carry to 0.
 - If the sum is more than 10, write the units digit of the sum directly below the two digits. Set the value of carry to 1.

3. Move to the next column of digits on the left. Add the two digits and the current value of carry.
 - If the sum is less than 10, write the sum directly below the two digits. Set the value of carry to 0.
 - If the sum is more than 10, write the units digit of the sum directly below the two digits. Set the value of carry to 1.
 4. Repeat Step 3 until there are no more digits on the left.
 5. If the value of carry is 1, write 1 to the left of the bottom row.
-

EXERCISE SET 11.1

1. Add two 4-digit numbers using the steps we have written down. Make sure you follow the steps precisely; do not perform any action that is not explicitly mentioned. Are you able to obtain the correct result?
2. What happens if you add a 5-digit number to a 3-digit number. Do our steps handle this situation correctly?
3. Why is it important to align the columns from right to left?
4. In Step 3, why cannot the value of carry be more than 1?
5. What happens if we do not include the fifth step in the algorithm above? Give examples where the algorithm will work correctly and where it will fail to work.

There is also an important assumption in this algorithm, namely, that we know how to add single digit numbers without counting them out. Typically, any algorithm is built using such basic steps that we assume we can perform directly.

The main advantage of this algorithm is that it is much faster than counting explicitly. For instance, if we move from adding 3-digit numbers to adding 4-digit numbers, we have 10 times as many dots to draw and count. On the other hand, if we add the digits right to left, as in the algorithm above, we only have to add 1 more column. Likewise, 5-digit numbers would generate 100 times as many dots as 3-digit numbers, but only 2 more columns to be added.

In concrete terms, suppose we can count one dot per second. To count the dots in $33 + 27$ would take a minute, to count $334 + 272$ would take about ten minutes, while $3347 + 2729$ would take over one hour and forty minutes.

On the other hand, suppose we work slowly and carefully and take 15 seconds to add two single digit numbers with a carry. Adding $33 + 27$ would take about half a minute, while $334 + 272$ would take about 45 seconds, while $3347 + 2729$ would take about one minute.

Whenever we provide an algorithm, we also have to justify why it works. Combining the dots and counting the sum is how addition is defined. Why does adding individual digits from right to left result in the sum of the two numbers? The key idea is that we are grouping each number into units, tens, hundreds, ... and adding each group separately. Whenever a smaller group, say units, generates a large group of ten, we carry over the newly formed ten to the tens group.

Think and Reflect

1. See if you can complete the argument about grouping by units, tens, hundreds, ... to justify why the addition algorithm works.
2. How would you modify the algorithm to add two decimal fractions?

In the rest of this chapter, we will explore the idea of an algorithm in more detail and in some other contexts.

11.2 GREATEST COMMON DIVISOR

Let us look at a familiar problem: computing the **greatest common divisor (gcd)** or **highest common factor (hcf)** of two numbers. We have already learned different ways to solve this problem, using prime factorisation and other methods.

How can we solve the problem from the definition of gcd? How do we write down an algorithm for this?

The first step is to understand what is required and break down the problem into smaller pieces that we know how to solve.

- We want the greatest common divisor of two numbers.
- For this, we need to first compute the divisors of both numbers and note them down.
- Once we have done this, we need to compare the two collections of divisors and find the largest number that is present in both collections.

11.2.1 Computing the Divisors of a Number

Suppose we want to compute $\text{gcd}(375, 825)$.

How do we find the divisors of 375? The divisors of a number are just the factors of that number. Clearly any divisor, or factor, must lie between 1 and 375. The simplest way to find the divisors is to check each number between 1 and 375 and see if it divides 375.

We also have to keep track of the divisors of 375 so that we can compare them with the divisors of 825. We can maintain each of these in a sequence, or a **list**. How do we create the list of divisors of a number? When we start, we do not know any divisors, so the list of divisors is initially empty. Every time we find a new divisor, we add it at the end of the list.

Here is how we can write down this algorithm to find the list of divisors of any number n .

Algorithm to find the divisors of n

1. Start with an empty list-of-divisors.
 2. For each number j in the sequence $1, 2, 3, \dots, n$
 - if j divides n , add j to the list-of-divisors.
-

Notice how we use names **list-of-divisors** and j to identify the values that we keep track of in this algorithm. If we do not use j to identify the number we are checking next as a possible divisor, it would be clumsy to describe what to do each time we find a divisor.

Running through an algorithm step by step is also called **executing** the algorithm. Let us execute this algorithm for a small number, say 18. We use square brackets to denote a list of values—for instance, $[1, 4, 9, 16]$.

Let us see how this algorithm works for a small number, say 18.

- Initially list-of-divisors is empty $[]$.
- 1 divides 18, so we update list-of-divisors to $[1]$.
- 2 divides 18, so we update list-of-divisors to $[1, 2]$.
- 3 divides 18, so we update list-of-divisors to $[1, 2, 3]$.
- 4 does not divide 18, so we leave list-of-divisors unchanged as $[1, 2, 3]$.
- 5 does not divide 18, so we leave list-of-divisors unchanged as $[1, 2, 3]$.

- 6 divides 18, so we update *list-of-divisors* to [1, 2, 3, 6].
- 7 and 8 do not divide 18, so we leave *list-of-divisors* unchanged as [1, 2, 3, 6].
- 9 divides 18, so we update *list-of-divisors* to [1, 2, 3, 6, 9].
- 10, 11, 12, ..., 17 do not divide 18, so we keep *list-of-divisors* unchanged as [1, 2, 3, 6, 9].
- 18 divides 18, so we update *list-of-divisors* to [1, 2, 3, 6, 9, 18].

Thus, at the end of the algorithm we have $\text{divisors}(18) = [1, 2, 3, 6, 9, 18]$. Notice that the list is also in increasing order, because we checked for divisors systematically, from 1 to 18.

What is the basic step that we assumed in this algorithm? It is that we know how to compute when a smaller number divides a larger number.

If we go through the steps of the algorithm, we get the following answers.

- $\text{divisors}(375) = [1, 3, 5, 15, 25, 75, 125, 375]$
- $\text{divisors}(825) = [1, 3, 5, 11, 15, 25, 33, 55, 75, 165, 275, 825]$

Think and Reflect

Try to execute the algorithm to compute the divisors of 15, 135, and 775. How does the amount of work increase as the numbers grow?

11.2.2 Finding the Greatest Common Divisor

The second step in our procedure to find the gcd is to compare the lists of divisors of the two numbers and find the largest number that appears in both lists.

If we look through the divisors of 375 and 825 above, we see that the answer is 75. What if the numbers were 54000 and 81000?

- $\text{divisors}(54000) = [1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 27, 30, 36, 40, 45, 48, 50, 54, 60, 72, 75, 80, 90, 100, 108, 120, 125, 135, 144, 150, 180, 200, 216, 225, 240, 250, 270, 300, 360, 375, 400, 432, 450, 500, 540, 600, 675, 720, 750, 900, 1000, 1080, 1125, 1200, 1350, 1500, 1800, 2000, 2160, 2250, 2700, 3000, 3375, 3600, 4500, 5400, 6000, 6750, 9000, 10800, 13500, 18000, 27000, 54000]$

- divisors(81000) = [1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 25, 27, 30, 36, 40, 45, 50, 54, 60, 72, 75, 81, 90, 100, 108, 120, 125, 135, 150, 162, 180, 200, 216, 225, 250, 270, 300, 324, 360, 375, 405, 450, 500, 540, 600, 648, 675, 750, 810, 900, 1000, 1080, 1125, 1350, 1500, 1620, 1800, 2025, 2250, 2700, 3000, 3240, 3375, 4050, 4500, 5400, 6750, 8100, 9000, 10125, 13500, 16200, 20250, 27000, 40500, 81000]

It is no longer so simple to just inspect the lists to find the largest element common to both of them! We need to do this systematically.

We compute a new list, the list of common divisors. As before, when we compute the list of divisors, this list is initially empty. For each divisor in the first list, we check if it belongs to the second list. If it does, we add it to the list of common divisors. Notice that 1 divides every number, so the list of common divisors will have at least one entry, 1.

Using this, we can compute the list of common divisors of 54000 and 81000, to be [1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 25, 27, 30, 36, 40, 45, 50, 54, 60, 72, 75, 90, 100, 108, 120, 125, 135, 150, 180, 200, 216, 225, 250, 270, 300, 360, 375, 450, 500, 540, 600, 675, 750, 900, 1000, 1080, 1125, 1350, 1500, 1800, 2250, 2700, 3000, 3375, 4500, 5400, 6750, 9000, 13500, 27000].

Since we know that each list of divisors is in increasing order, it must be the case that the list of common divisors is also in increasing order. Hence the largest common divisor is the rightmost element in the list. In other words the gcd of 54000 and 81000 is 27000.

11.2.3 First Algorithm for gcd

Combining the steps above, we can write out a complete algorithm to compute the gcd of two numbers m and n . This makes use of the earlier algorithm to compute the list of divisors of a number.

Algorithm to find gcd(m, n)

1. Let `divisors-of-m` be the list obtained by computing `divisors(m)`
2. Let `divisors-of-n` be the list obtained by computing `divisors(n)`
3. Start with an empty list `common-divisors`

4. For each number x in the list of divisors-of- m
 - If x also appears in the list of divisors-of- n , add x to the list common-divisors
5. Report the rightmost element of the list common-divisors as $\text{gcd}(m, n)$

When we execute this algorithm for $\text{gcd}(375, 825)$ we check whether each of the divisors $[1, 3, 5, 15, 25, 75, 125, 375]$ of 375 appears in the list of divisors $[1, 3, 5, 11, 15, 25, 33, 55, 75, 165, 275, 825]$ of 825. We see that 1, 3, 5, 15, 25 and 75 are in the second list while 125 and 375 are not. So we compute the list of common divisors to be $[1, 3, 5, 15, 25, 75]$. Since 75 is the rightmost element in this list, $\text{gcd}(375, 825) = 75$.

11.3 DATA STRUCTURES

When we execute an algorithm, we compute intermediate quantities, such as the list *divisors-of- m* and the list *divisors-of- n* . To keep track of these values, we use names such as *divisors-of- m* and *divisors-of- n* , respectively, so that we can refer to them later in the algorithm. As we remarked earlier, assigning names to intermediate values is important for writing down algorithms concisely.

Also, these intermediate values are typically more complicated than just simple numbers. In this case, we maintain lists of numbers and, in fact, lists that are arranged in ascending order. A **list** is an example of a **data structure**. Data structures allow us to organise information in a manner that helps make an algorithm more efficient. We will explore this idea in higher grades.

EXERCISE SET 11.2

1. Suppose List 1 and List 2 are two lists of numbers in increasing order.
 - (i) Write an algorithm to find elements in List 1 that are not present in List 2.
 - (ii) Write an algorithm to find elements in List 2 that are not present in List 1.
2. Describe an algorithm to compute the least common multiple (lcm) of two numbers.

- *3. Divisors occur in pairs. For instance, the divisors of 18 are (1, 18), (2, 9) and (3, 6).
- (i) If we write out divisors in pairs, how many numbers do we have to examine between 1 and n to find all the divisors of n ?
 - (ii) If we list out the divisors in pairs, will our gcd algorithm still work in the manner we have described?

11.3.1 Improving the Algorithm

Compute both lists of divisors together

In the algorithm above, we first check all numbers between 1 and m to find the divisors of m . Then we start again with 1 and check all numbers between 1 and n to find the divisors of n . This means we have to scan $m + n$ possible divisors.

We can combine these two scans into one. Let $\max(m, n)$ be the maximum of m and n . We run through each j from 1 to $\max(m, n)$.

- If j divides m we add j to the list of divisors of m .
- If j divides n we add j to the list of divisors of n .

This reduces the number of possible divisors to be checked from $m + n$ to $\max(m, n)$.

Directly compute the list of common divisors

If we combine the scans, we can directly compute the list of common divisors, instead of first computing the divisors of each number and then combining the lists.

- For each number j from 1 to $\max(m, n)$, if j divides both m and n , add j to the list of common divisors.

In fact, a common divisor cannot be larger than the smaller of the two numbers, so we can check for common divisors up to the minimum of m and n rather than the maximum.

- For each number j from 1 to $\min(m, n)$, if j divides both m and n , add j to the list of common divisors.

Do away with lists

Do we need to maintain the list of all common divisors? Each time we find a larger common divisor, the ones that we have found earlier are useless!

Let us keep track of a quantity called most-recent-common-divisor. We know that 1 is always a common divisor, so we can start with most-recent-common-divisor set to 1. We then scan all numbers from 2 to $\min(m, n)$.

- For each k from 2 to $\min(m, n)$, if k divides both m and n , update most-recent-common-divisor to k .

At the end of this scan, most-recent-common-divisor directly gives us the gcd.

Example: Let $m = 6, n = 12$.

- $\min(6, 12) = 6$
 - Initially set most-recent-common-divisor to 1
 - $k = 2$ divides both 6 and 12.
Update: most-recent-common-divisor to 2.
 - $k = 3$ divides both 6 and 12.
Update: most-recent-common-divisor to 3.
 - $k = 4$ does not divide 6.
most-recent-common-divisor remains 3.
 - $k = 5$ does not divide 6.
most-recent-common-divisor remains 3.
 - $k = 6$ divides both 6 and 12.
Update: most-recent-common-divisor to 6.
- Scan complete. $\gcd(6, 12) = 6$

EXERCISE SET 11.3

1. How would our original algorithm change if we computed the divisors of n by examining the numbers from 1 to n in reverse order, from n down to 1?
2. What about the last algorithm described above? What happens when we look at common divisors starting from $\min(m, n)$ and work backwards to 1?

11.3.2 Analysing these Algorithms

We started with a very basic algorithm. Since this was derived directly from the definition of gcd, it is easy to justify that the result it computes is actually the gcd.

Each successive refinement that we have proposed improves the efficiency but does not substantially change what is being computed. Hence, we can argue that all of the algorithms we have seen so far compute the correct value.

How efficient are these algorithms? Even in the final version, where we eliminated lists, we had to examine all numbers from 1 to $\min(m, n)$. This is similar to what we saw when adding numbers by counting. If the smaller number goes from 3 digits to 4 digits, we have to scan 10 times as many values to find the common divisors. If we go from 3 digits to 5 digits, the work grows by a factor of 100.

As we did for adding, can we find an algorithm for gcd where the effort is proportional to the number of digits in the number, rather than the value of the number?

11.3.3 Euclid's Subtraction Algorithm

An elegant solution to achieve this was written down by Euclid, in his book *Elements*. Historians believe that other Greek mathematicians knew of this algorithm before Euclid, but he was the first to put it down in writing.

Reducing a problem to a simpler one

The original version of Euclid's algorithm makes use of the following observation. Let us assume that $m \geq n$. Then, $\gcd(m, n)$ is the same as $\gcd(n, m - n)$.

Why is this the case?

- Let d be a common divisor of m and n . Then we can write m as ad and n as bd , where a and b are natural numbers. This means that $m - n = ad - bd = (a - b)d$, so d is also a divisor of $m - n$.
- On the other hand, if d divides both n and $m - n$, then $n = xd$ and $m - n = yd$ for some natural numbers x and y . This means that $n + (m - n) = m = xd + yd = (x + y)d$, so d also divides m .

This is shown visually below. If we can 'tile' m and n using blocks of size d , we can also tile $m - n$ by blocks of the same size. In fact, Euclid used a similar pictorial proof to justify his algorithm.

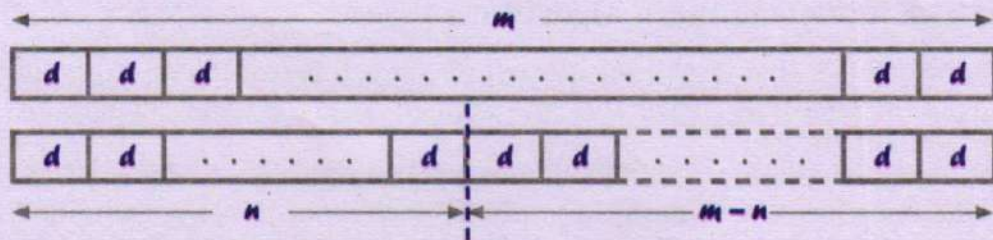


Fig. 11.1: Justification for Euclid's gcd algorithm

This observation tells us that any common divisor of m and n is also a common divisor of n and $m - n$, and the converse is also true. Hence the set of common divisors are the same in both cases, which means that $\gcd(m, n) = \gcd(n, m - n)$.

Euclid's algorithm reduces the problem of finding $\gcd(m, n)$ to that of finding $\gcd(n, m - n)$. The second problem involves smaller numbers, so it should be a simpler problem to solve.

How do we solve it? We apply the same strategy: reduce it to $\text{gcd}(x, y)$ where $x = \min(n, m - n)$ and y is the difference between n and $m - n$.

When do we stop? Suppose we compute the gcd of a number with itself, say $\text{gcd}(23, 23)$. The answer should clearly be 23. When we reduce this we get $\text{gcd}(23, 0)$. In other words, if we reach $\text{gcd}(m, 0)$, the answer is m .

Euclid's algorithm for $\text{gcd}(m, n)$

1. If $m < n$, reverse the numbers and compute $\text{gcd}(n, m)$
 2. If $n = 0$, report the answer as m
 3. Otherwise, reduce the problem to compute $\text{gcd}(n, m - n)$
-

Let us apply this algorithm to $\text{gcd}(375, 825)$, the problem that we saw before.

- Since $375 < 825$, Step 1 asks us to reverse the numbers and compute $\text{gcd}(825, 375)$.
- Since $825 - 375 = 450$, Step 3 asks us to reduce the problem to computing $\text{gcd}(375, 450)$.
- Since $375 < 450$, Step 1 asks us to reverse the numbers and compute $\text{gcd}(450, 375)$.
- Step 3 asks us to reduce the problem to computing $\text{gcd}(375, 75)$.
- Step 3 asks us to reduce the problem to computing $\text{gcd}(75, 300)$.
- Since $75 < 300$, Step 1 asks us to reverse the numbers and compute $\text{gcd}(300, 75)$.
- Step 3 asks us to reduce the problem to computing $\text{gcd}(75, 225)$.
- Since $75 < 225$, Step 1 asks us to reverse the numbers and compute $\text{gcd}(225, 75)$.
- Step 3 asks us to reduce the problem to computing $\text{gcd}(75, 150)$.
- Since $75 < 150$, Step 1 asks us to reverse the numbers and compute $\text{gcd}(150, 75)$.
- Step 3 asks us to reduce the problem to computing $\text{gcd}(75, 75)$.
- Step 3 asks us to reduce the problem to computing $\text{gcd}(75, 0)$.
- Since $n = 0$, Step 2 tells us the answer is 75.

In the example above, we reduced the problem 7 times before finding the answer. Are we close to our original goal of finding an algorithm where the effort is proportional to the number of digits? Unfortunately, no.

Consider $\gcd(99, 2)$. This will reduce successively to $\gcd(97, 2)$, $\gcd(95, 2)$, $\gcd(93, 2)$, ..., $\gcd(5, 2)$, $\gcd(3, 2)$, $\gcd(2, 1)$, $\gcd(1, 1)$, and $\gcd(1, 0)$ eventually giving us the answer 1. In general, if we start with an odd number $2k + 1$, $\gcd(2k + 1, 2)$ will require about k reductions before we get the answer 1. This means the number of steps is proportional to the value we start with, rather than the number of digits used to write down the value.

11.3.4 Āryabhaṭa's Division Algorithm

In Indian treatises like Āryabhaṭa's Āryabhaṭīya (499 CE), we see the "long division" method for finding GCD using Indian place value numerals. It makes an efficient use of repeated "division by remainder" in place of repeated subtraction. For example, when one starts with 99 and subtracts 2 repeatedly, what remains, once one cannot subtract 2 anymore, is the remainder when 99 is divided by 2.

Hence, to improve the performance of Euclid's algorithm, we can change the reduction step: instead of reducing $\gcd(m, n)$ to $\gcd(n, m - n)$, we reduce it to $\gcd(n, m \bmod n)$, where $m \bmod n$ stands for the remainder when m is divided by n . For example, $17 \bmod 5$ is 2 and $33 \bmod 7$ is 5.

We saw earlier that a divisor d 'tiles' both m and n if and only if it 'tiles' both n and $m - n$. In a similar way, we can argue that d 'tiles' m and n if and only if it tiles m and $m \bmod n$. This justifies reducing $\gcd(m, n)$ to $\gcd(n, m \bmod n)$.

Here is the modified algorithm. Notice that the only change is in the third statement. The stopping condition is the same as before.

-
1. If $m < n$, reverse the numbers and compute $\gcd(n, m)$
 2. If $n = 0$, report the answer as m
 3. Otherwise, reduce the problem to computing $\gcd(n, m \bmod n)$
-

Consider the earlier example $\gcd(99, 2)$ that took a long time to converge. Here is what happens when we execute the improved algorithm.

- Step 3 asks us to reduce the problem to computing $\gcd(2, 1)$.
- Step 3 asks us to reduce the problem to computing $\gcd(1, 0)$.
- Since $n = 0$, Step 2 tells us the answer is 1.

In other words, we needed only two reduction steps to get to the final answer. Check that this would happen for any similar problem of the form $\gcd(2k + 1, 2)$.

The above modified algorithm can be carried out concisely using the “long division” method given by Indians at least from the time of Āryabhaṭa. We illustrate the method through the following examples—

$$\begin{array}{r}
 \text{1. gcd}(99, 2) \quad 2 \overline{)99} \begin{array}{l} 49 \\ 98 \\ \hline 1 \end{array} 2 \begin{array}{l} 2 \\ 2 \\ \hline 0 \end{array} \\
 \begin{array}{c} \parallel \\ \text{gcd}(2, 1) \end{array} \quad \begin{array}{c} \nearrow \\ \text{gcd} \end{array}
 \end{array}$$

$$\begin{array}{r}
 \text{2. gcd}(825, 375) \quad 375 \overline{)825} \begin{array}{l} 2 \\ 750 \\ \hline 75 \end{array} 375 \begin{array}{l} 5 \\ 375 \\ \hline 0 \end{array} \\
 \begin{array}{c} \parallel \\ \text{gcd}(375, 75) \end{array} \quad \begin{array}{c} \nearrow \\ \text{gcd} \end{array}
 \end{array}$$

Here is another example—

$$\begin{array}{r}
 \text{3. gcd}(60, 16) \quad 16 \overline{)60} \begin{array}{l} 3 \\ 48 \\ \hline 12 \end{array} 16 \begin{array}{l} 1 \\ 12 \\ \hline 4 \end{array} 12 \begin{array}{l} 3 \\ 12 \\ \hline 0 \end{array} \\
 \begin{array}{c} \parallel \\ \text{gcd}(16, 12) \\ \parallel \\ \text{gcd}(12, 4) \end{array} \quad \begin{array}{c} \nearrow \\ \text{gcd} \end{array}
 \end{array}$$

In fact, it can be shown that the number of reductions needed in Āryabhaṭa’s division algorithm is now proportional to the number of digits in m and n . Unfortunately, the justification for this fact is too involved to present here, but you will learn about it in future grades.

A Pinch of History

The influential Persian mathematician Al-Khwārizmī (780–850 CE), a scholar at the “House of Wisdom” in Baghdad, developed a great liking for Indian mathematics. He learnt Sanskrit, and around 820 CE wrote a detailed treatise expounding the Indian place value number system and the Indian procedures for the four fundamental operations — addition, subtraction, multiplication, and division. His original is lost, but its 12th-century Latin translation, *Liber Algorismi de numero Indorum* (“The Book of Al-Khwārizmī on Indian Numerals”), survives and played a major role in spreading Indian arithmetic through medieval Europe. Al-Khwārizmī’s name became so closely tied to these methods that the word *algorismus* — from *Algoritmi*, the Latin form of his name — came to mean any Indian method for numerical computation; it later contracted to *algorism* and then *algorithm*. Etymologically, then, “algorithm” originally denoted an Indian method of arithmetic, before it broadened to mean any well-defined, step-by-step systematic procedure for solving a class of problems.

END-OF-CHAPTER EXERCISES

1. Compute the following using the improved version of Euclid's algorithm.
 - (i) $\gcd(375, 825)$
 - (ii) $\gcd(51000, 81000)$
 - (iii) $\gcd(1789287, 237656)$
 - (iv) $\gcd(2587392, 157656)$
2. Assume that $m \geq n$. Verify that d divides m and n if and only if d divides both n and $m \bmod n$.
3. Write an algorithm $\text{prime}(n)$ to check if n is prime.
(Hint: A prime number p has exactly two distinct factors, 1 and p . Can you make use of $\text{divisors}(n)$ to write out $\text{prime}(n)$?)
- *4. Write an algorithm $\text{primedivisors}(n)$ to compute the list of divisors of n that are prime numbers.
(Hint: Compute $\text{divisors}(n)$ and then filter out the primes in this list.)
- *5. We can also find the $\gcd$ of two numbers by computing prime factorisation of both the numbers. Try to write an algorithm to compute the prime factorisation of a number.
 - (i) The prime factorisation of 180 is $2^2 \times 3^2 \times 5^1$. How would you represent this?
 - (ii) How would you compare the prime factorisations of two numbers?

CHAPTER SUMMARY

- An **algorithm** is a systematic procedure to solve a problem.
- Algorithms consist of basic steps that we know how to execute.
- We may execute the same step a number of times; for instance, for each j in $1, 2, \dots, n$, check if j is a divisor of n .
- A step in an algorithm may be conditional; for instance, add j to the list of divisors of n only if j divides n .
- When we describe an algorithm to solve a problem, we need to justify that it is correct.
- We can also analyse how many steps an algorithm takes, in terms of the size of its input, and decide how efficient it is.



Quadrilaterals, which we also call 4-gons, occur everywhere in daily life. For example, we often see flooring made of square tiles. We say that the floor has been ‘tiled’ using those squares.

Think and Reflect

Can we use any given quadrilateral to tile the plane? If not, which quadrilaterals can be used and which cannot?

Recall that **tiling** the plane means covering it with copies of the given shape or shapes without gaps or overlaps. Tiling the plane with a rectangle is easy, but it seems difficult with an irregular quadrilateral. We will answer the above question at the end of the chapter (can you do it now?) and understand the answer using what we learn in this chapter. However, to start with, we will discuss a simple new example of a tiling.

Imagine that tiling with rectangles is done using a grid of two sets of long wooden sticks, one set horizontal and the other set vertical. See Fig. 12.1A.

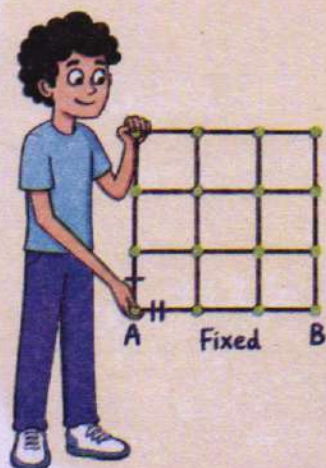


Fig. 12.1A

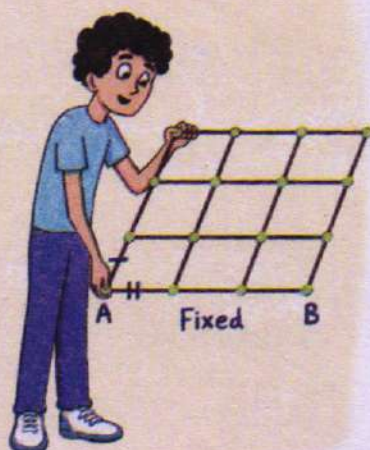


Fig. 12.1B

These sticks are pinned together at each intersection point such that they are free to rotate about those points. Now suppose you pick up this grid. You hold the bottom stick fixed with one hand and push the vertical side with the other. See Fig. 12.1B. Each set of parallel lines will stay parallel but the angle between the two sets will no longer be 90° . Thus we see that the plane can be tiled using copies of any fixed parallelogram.

In this chapter, we will ask and answer several questions about quadrilaterals, paying particular attention to parallelograms. We will investigate their properties, examine ways to recognise them, and see important applications in geometry. At the end, we will answer the question above about whether one can tile the plane with a given quadrilateral.

12.1 WHAT EXACTLY IS A QUADRILATERAL?

Think and Reflect

Informally, a quadrilateral is a figure with four straight sides, as in the first figure ABCD in Fig. 12.2 below. But consider the other six figures in Fig. 12.2: the five plane figures NOPE, SILY, DART, CUTS, OPENS and the non-planar BENT. Should we call all these figures quadrilaterals? If you answer 'no' for any of them, how will you define a quadrilateral so that such a figure is excluded? As you can see, some care is needed to precisely define what we think of as a quadrilateral.

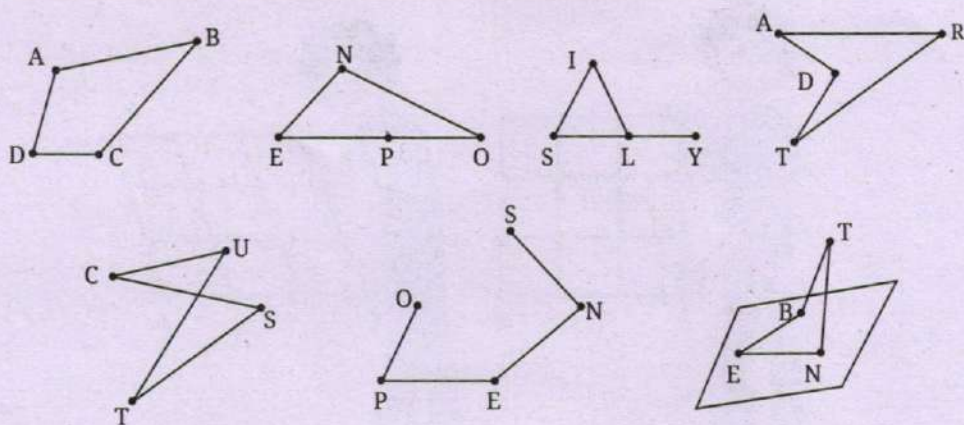


Fig. 12.2

To prepare, let us first consider how we can define a triangle. Let A , B and C be three points. Can we say that $\triangle ABC$ consists of points on the three line segments AB , BC , CA ? Yes, provided A , B , C are not collinear. Otherwise, we just get a line segment: see Fig. 12.3.

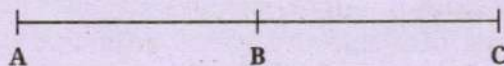


Fig. 12.3: Not a triangle

When A , B , C are non-collinear, we call them the **vertices** of $\triangle ABC$ and the three segments AB , BC and CA the **sides** of $\triangle ABC$.

Think and Reflect

Can we similarly define a quadrilateral $ABCD$?

Take any 4 distinct non-collinear points A , B , C and D . Will segments AB , BC , CD and DA always form a quadrilateral as we visualise it? See figures NOPE, SILY, DART, CUTS, OPENS and BENT in Fig. 12.2 again. Do you see, OPENS that there are some subtle new issues to consider?

First, to rule out examples like NOPE and SILY, no three vertices should be collinear. Otherwise, the two adjacent sides formed by the three collinear vertices will be along the same line, giving a triangle like NOPE or a figure with overlapping sides like in SILY.

Second, to rule out examples like BENT, all four vertices should be in the same plane. Four-sided figures like BENT are called **non-planar** quadrilaterals. We will study only planar quadrilaterals (except in a few exercises).

Third, do we want to call CUTS a quadrilateral? The answer depends on what we decide! Ordinarily, CUTS is not considered a quadrilateral. If we want to include it, we call it a self-intersecting quadrilateral.

Can you come up with a definition that rules out self-intersecting quadrilaterals like CUTS? Answer: we should require that all the points of the quadrilateral, other than the vertices, must lie on exactly one side.

Can you see how to rule out figures like OPENS?

Putting together all three requirements, we get the following definition.

Definition 1. Suppose A , B , C , D are four distinct points in a plane. Then the points on the 4 segments AB , BC , CD and DA are said to form **quadrilateral** $ABCD$ if every such point other than A , B , C , and D lies on exactly one of these four segments.

Now that we have precisely defined a quadrilateral, let us give similarly precise definitions of familiar terms associated with a quadrilateral.

A, B, C, D are called the **vertices** of quadrilateral ABCD; segments AB, BC, CD, DA are its **sides** or **edges**; segments AC and BD are its **diagonals**. The two endpoints of a side are called **adjacent vertices**. Two sides with a common endpoint are called **adjacent sides**. A quadrilateral has 4 **internal angles**, each formed at a vertex by a pair of adjacent sides. Internal angles at adjacent vertices are called a pair of **adjacent angles**. How will you define **opposite sides**? **Opposite angles**? See Exercise Set 12.1, Q1.

Note that quadrilateral ABCD can also be denoted as BCDA, CDAB, DABC, DCBA, ADCB, BADC or CBAD but not by any other sequence of vertices. (Draw ABCD and trace the vertices in various orders to see why.)

When we refer to a quadrilateral, we always mean a planar non-self-intersecting quadrilateral, unless stated otherwise.

Convex Quadrilaterals

Should DART in Fig. 12.2 be considered a quadrilateral? It seems different from our usual mental picture of a quadrilateral because it has a dent, as if someone has taken a bite out of triangle ART! We call DART a *non-convex* quadrilateral.

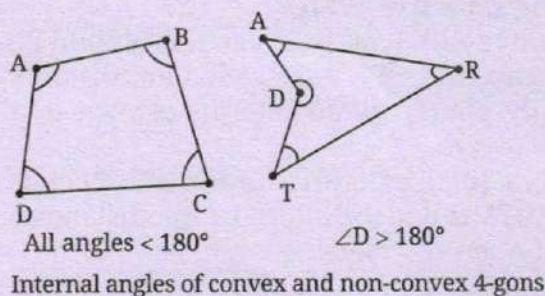


Fig. 12.4

A convex quadrilateral is one without a dent. How can we define this precisely? We can see in DART that at the dent at vertex D, the internal angle is more than 180° . So we call a quadrilateral *convex* when *all* of its internal angles are less than 180° .

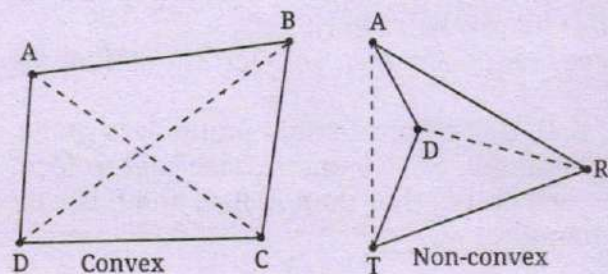
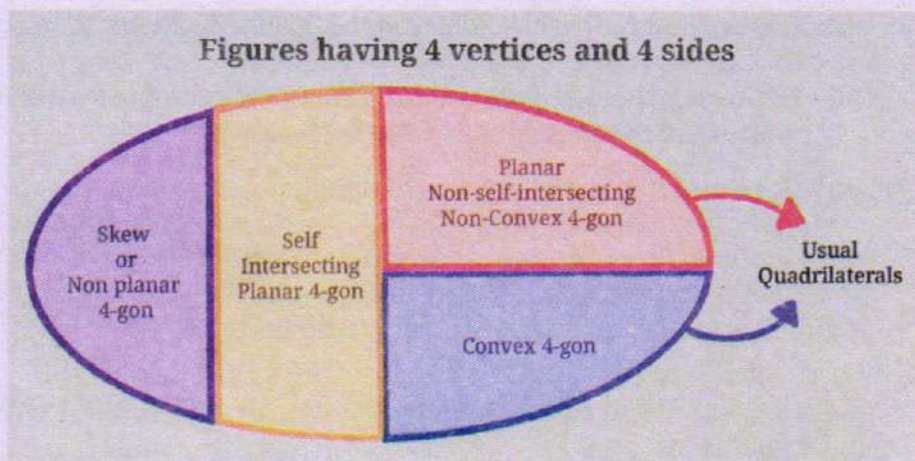


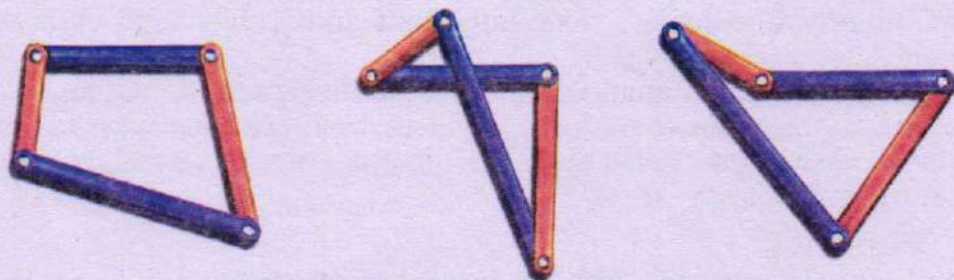
Fig. 12.5

There are other ways of testing convexity. For example, visually verify that the diagonals of a convex quadrilateral intersect while those of a non-convex quadrilateral don't intersect.



Venn Diagram Representation

4-gons that do not fit your mental image of a quadrilateral may look strange. Do we really need to care about, e.g., self-intersecting 4-gons? Yes, because they do occur in real life, such as in 4-bar linkages. Linkages are widely used mechanical devices with several interlinked bars. Some ends of these bars are fixed, while other interlinked ends are free to rotate around the fixed ones. Search the internet for a video showing how linkages move. Linkages can attain a range of configurations, including non-convex and self-intersecting figures, as shown below. So engineers have to take such figures into account.



EXERCISE SET 12.1

1. Let $ABCD$ be a quadrilateral.
 - (i) List all sides of $ABCD$ adjacent to side AB . List all sides opposite to AB .

- (ii) List all angles of ABCD adjacent to $\angle A$. List all angles opposite to $\angle A$.
 - (iii) Define a pair of opposite sides and a pair of opposite angles without using the names of the vertices.
- *2. You have used internal angles of quadrilaterals, but they too require an exact definition, just like how we gave one for a quadrilateral. Precisely define the internal angle of a quadrilateral at a given vertex. Your answer should work for a non-convex quadrilateral too. (**Hint:** use the opposite vertex as well.)
3. In a quadrilateral ABCD, suppose $AB \parallel DC$. Can ABCD be non-convex? What if we instead assume $AB = CD$? What if we instead assume $\angle A = \angle C$?
4. Consider three non-collinear points A, B, C and draw the lines AB, BC, CA. For every possible location of point D in the plane outside these lines, decide if ABCD is self-intersecting, non-convex, or convex. (**Hint:** the three lines divide the plane into 7 regions.)
5. Can a quadrilateral be both self-intersecting and non-planar?

12.2 PARALLELOGRAMS

As we studied in Grade 8, parallelograms are quadrilaterals whose opposite sides are parallel. They include rhombuses, rectangles and squares and form a very important and widely used class of quadrilaterals.

To test if a given quadrilateral is a parallelogram, do we have to check that the opposite sides are parallel? Are there other ways to test this? To answer this, recall the following properties of a parallelogram that we proved last year.

Theorem 1:

- (a) *The opposite sides of a parallelogram are equal.*
- (b) *The opposite angles of a parallelogram are equal.*
- (c) *The diagonals of a parallelogram bisect each other.*

It is usually interesting to examine if converses of known theorems are true. If the converse of any of the three properties is true, it can be used as an alternate way to show that a quadrilateral is a parallelogram. To explore this, let us write the converses. For this, we first write (a), (b) and (c) in "If ... then ..." form.

- (a) If a quadrilateral is a parallelogram, then its opposite sides are equal.

Converse: If the opposite sides of a quadrilateral are equal, then it is a parallelogram.

Is this converse statement true?

- (b) If a quadrilateral is a parallelogram, then the opposite angles are equal.

Converse: If the opposite angles of a quadrilateral are equal, then it is a parallelogram.

Is this converse statement true?

- (c) If a quadrilateral is a parallelogram, then its diagonals bisect each other.

Converse: If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

Is this converse statement true?

Can you experiment and guess what the answers are?

It turns out that all three converse statements are true. To demonstrate this, it is useful to recall how we had proved Theorem 1:

Proof of Theorem 1:

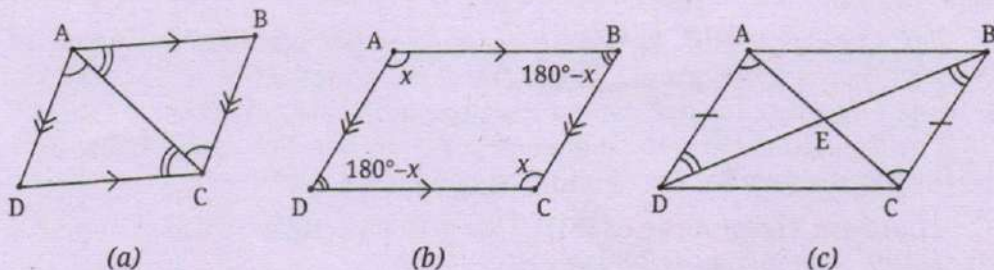


Fig. 12.6

Proof of (a): In the parallelogram ABCD, AC is a transversal for $AB \parallel DC$ and also for $AD \parallel BC$. So we get $\triangle ACD \cong \triangle CAB$ by the ASA test. Therefore, $AB = DC$ and $AD = BC$.

Proof of (b): $BC \parallel AD$ with AB a transversal. Therefore $\angle A + \angle B = 180^\circ$ as they are interior angles on the same side of this transversal. By similar reasoning, any two adjacent angles add up to 180° . By using algebra, $\angle A = \angle C$ and $\angle B = \angle D$.

Proof of (c): Call the intersection point of the two diagonals E. Using property (a), $\triangle AED \cong \triangle CEB$, by ASA or AAS. So $EA = EC$ and $EB = ED$, which means that E is the midpoint of both diagonals.

We will now deduce the converse statements. To prove the converse of a known theorem, we can try to run the original proof backwards. This strategy may or may not work. (For example, the reasoning “If $x = y$, then $x^2 = y^2$ ” cannot be reversed, because x and y may have opposite signs, e.g., we could have $x = 3$ and $y = -3$.) When the strategy does work, we should be careful to give proper justification for each step in the backward direction.

Theorem 2 (converse of (a)): *If the opposite sides of a quadrilateral are of equal length, then it is a parallelogram.*

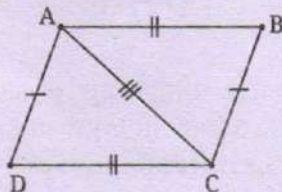


Fig. 12.7: Converse to (a)

Proof: Let ABCD be this quadrilateral. $\triangle ACD \cong \triangle CAB$ by SSS, so corresponding angles are equal. Using transversal AC, we get $AB \parallel DC$ and $AD \parallel BC$.

Pay close attention to how the above proof retraces the proof of (a) backwards. Both proofs go via the same congruence $\triangle ACD \cong \triangle CAB$, but the two proofs use different tests to justify the congruence (ASA vs SSS) and draw different conclusions. Try to use the same ‘backward retracing’ strategy for the remaining converses.

Theorem 3 (converse of (b)): *If the opposite angles of a quadrilateral are equal, then it is a parallelogram.*

Proof: Let ABCD be this quadrilateral. Because $\angle A = \angle C$ and $\angle B = \angle D$, we have

$$360^\circ = \angle A + \angle B + \angle C + \angle D = 2(\angle A + \angle B) = 2(\angle B + \angle C).$$

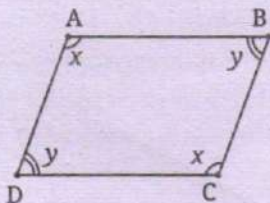


Fig. 12.8: Converse to (b)

$$2(x + y) = 360^\circ$$

Therefore $\angle A + \angle B = \angle B + \angle C = 180^\circ$. As in the proof of (b), we use the fact that AB is a transversal for the pair BC and AD. By $\angle A + \angle B = 180^\circ$ we deduce $BC \parallel AD$. Similarly, we get $AB \parallel DC$ by using $\angle B + \angle C = 180^\circ$.

Think and Reflect

The angles of a quadrilateral add up to 360° . Therefore, if the opposite angles are equal, what can we say about adjacent angles? Is the converse of your answer true? Conclude that Theorem 3 can also be stated as follows. "If each pair of adjacent angles in a quadrilateral ABCD...then ABCD is a parallelogram." Fill in the blank.

Theorem 4 (converse of (c)): A quadrilateral whose diagonals bisect each other is a parallelogram.

Proof: Let ABCD be this quadrilateral. Let E be the intersection point of the diagonals AC and BD. As $EA = EC$ and $EB = ED$, we have $\triangle AED \cong \triangle CEB$ by SAS. Now using $\angle DAE = \angle ECB$ along with transversal AC, we get $AD \parallel BC$. Similarly, $\triangle EAB \cong \triangle ECD$ gives $AB \parallel DC$.

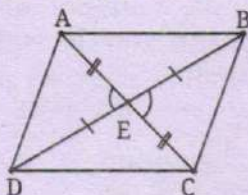


Fig. 12.9: Converse to (c)

Finally, there is one more useful test to check if a given quadrilateral ABCD is a parallelogram.

Theorem 5 (equal parallel sides test): A quadrilateral with one pair of equal and parallel opposite sides is a parallelogram.

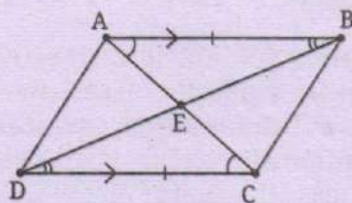


Fig. 12.10

Proof: Suppose in a quadrilateral ABCD we have $AB \parallel DC$ and $AB = DC$. Let E be the intersection point of the diagonals AC and BD. (Why must the diagonals intersect? See Exercise Set 12.1, Q3.) We have $\triangle EAB \cong \triangle ECD$ by ASA. So the diagonals bisect each other. By the converse of (c), ABCD is a parallelogram.

EXERCISE SET 12.2

- True or false?
 - A parallelogram with a right angle is a rectangle.
 - A rhombus with perpendicular diagonals is a square.
 - If the diagonals of a parallelogram are equal, then it is a rectangle.
- The diagonal AC of a parallelogram ABCD bisects $\angle A$. Show that it also bisects $\angle C$ and that ABCD is a rhombus.
- The following questions examine converses of true properties. Answer them with Yes or No. If your answer is No, what extra condition can you add so that the answer becomes Yes?
 - If the diagonals of a quadrilateral ABCD bisect its angles, must ABCD be a rhombus?
 - If the diagonals of a quadrilateral ABCD bisect each other at right angles, must ABCD be a rhombus?
 - If the diagonals of a quadrilateral ABCD are of equal length, must ABCD be a rectangle? (In the ancient Indian study of quadrilaterals, equality of diagonals was considered significant. Quadrilaterals were first classified according to whether their diagonals were equal or not, before considering equality of sides.)
- Let ABCD be a parallelogram with $AB \neq BC$. Show that the pairwise intersection points of the four angle bisectors form the vertices of a rectangle (see Fig. 12.11). Why did we assume $AB \neq BC$?

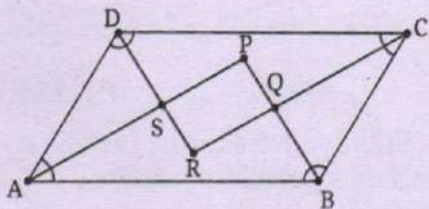
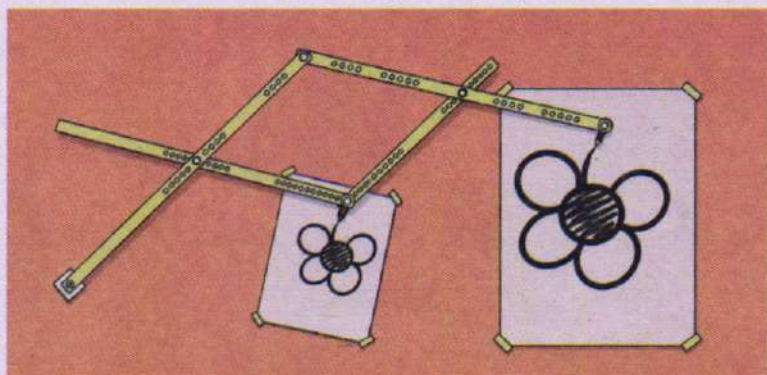


Fig. 12.11

12.3 APPLICATIONS OF PARALLELOGRAMS

We saw that parallelograms have several nice geometric properties. These properties make the parallelogram a versatile shape with many real-world applications. In physics, a fundamental law uses parallelograms to combine certain physical quantities like forces, as you will learn in a later year. Parallelograms are used in several mechanical linkages, such as the parallel motion linkage discovered by James Watt,

which formed an important part of his steam engine. His design used the concept behind another parallelogram-based device called a pantograph, which can make exact scaled copies of a drawing: search the internet for a video showing how a pantograph works. It is ingenious!



Schematic drawing of a pantograph

Parallelograms also serve as a powerful tool within geometry. Let us illustrate this by using parallelograms in a clever way to prove an important fact about triangles. We will discover this fact through exploration.

12.3.1 An Important Property of Triangles

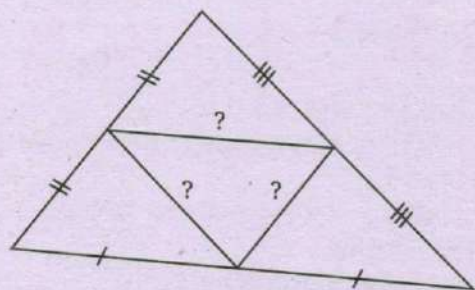


Fig. 12.12

On a piece of paper, draw and then cut out two copies of the same triangle. Keep one copy aside. On the other, mark the midpoint of each side. (You can do this by folding the paper to join two vertices at a time.) Draw the triangle made by the three midpoints and cut the paper along each of these lines. Now you have four smaller triangles. Compare them to each other and to the intact copy of the original triangle. What do you notice?

It appears that all four smaller triangles are congruent to each other! And each seems to be a smaller copy of the original triangle. Verify that this pattern holds for triangles of other shapes.

Think and Reflect

Try to prove directly that the four smaller triangles are congruent. You will see that no congruence test applies, because many quantities are unknown.

Here is one idea. The difficulty might be because we are considering too many new things at once: three new points and three new segments. Can we try to break this down into steps?

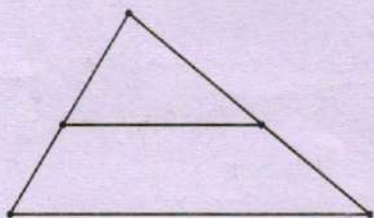


Fig. 12.13

Let us see what happens with just one of the three new segments. Can you make a guess about this new segment, specifically about its direction and length compared with the unused third side? The triangles you cut earlier will help. It appears that the segment is parallel to the third side and has half the length! Let us try to justify this using reasoning.

Geometric reasoning: To set up, suppose that in $\triangle ABC$, P and Q are the midpoints of side AB and AC respectively. Now comes the key idea. Construct the line l through C parallel to BA . Suppose it intersects line PQ at point R .

(See the comment about this line immediately after the statement of Theorem 6.)

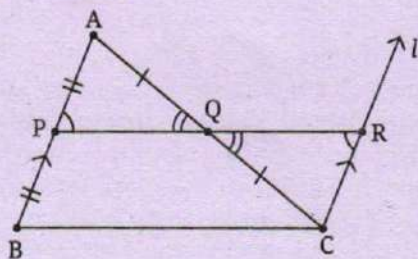


Fig. 12.14

In $\triangle APQ$ and $\triangle CRQ$, we have $AQ = CQ$,
 $\angle AQP = \angle CQR$ (vertically opposite angles), and
 $\angle APQ = \angle CRQ$ (since $AP \parallel RC$ by construction).

So $\triangle APQ \cong \triangle CRQ$ by AAS.

Therefore $PQ = QR$ (by congruence) = $\frac{PR}{2}$ and $CR = AP$. Combining $CR = AP$ with $AP = BP$, we get $CR = BP$. Hence by Theorem 5

(equal parallel sides test), BCRP is a parallelogram. This gives $PQ \parallel BC$ and also $PQ = \frac{PR}{2} = \frac{BC}{2}$. We have deduced the following important fact.

Theorem 6 (Midpoint Theorem): *The segment joining the midpoints of two sides of a triangle is parallel to the third side and has half the length.*

In the reasoning for this theorem, the idea to take a completely new line l appeared out of nowhere, and worked like magic. What is your reaction to this? Do you feel a sense of wonder? Do you perhaps feel discouraged because it seems impossible to think of such an idea on your own? This type of auxiliary construction is a classic geometric trick that one learns with experience, so no need to be disheartened! See Exercise 15 at the end of the chapter for another proof that uses a different, slightly less magical construction.

Can you now prove the original claim about cutting $\triangle ABC$ into four congruent parts by using midpoints of all three sides? You are asked to do this in Exercise Set 12.3 Q1 below.

Think and Reflect

Read the statement of the Midpoint Theorem again. The following questions seem natural to ask. What can we say about (1) a segment PQ (with P on side AB and Q on side AC) that is parallel to the third side and has half the length, and (2) the line that passes through the midpoint of one side and is parallel to another side? We will address question (1) in Exercise 22 at the end of the chapter. For now, experiment and see if you can make a guess about question (2). It appears that this line must bisect the remaining side! Let us try to justify this.

Geometric reasoning: To set up, take $\triangle ABC$ and suppose the line through midpoint P of AB , drawn parallel to BC , meets AC at point Q . Again draw the line parallel to BA through C , intersecting line PQ at point R . Tracing the proof of the Midpoint Theorem backwards works easily as follows.

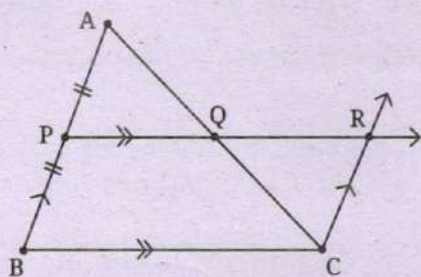


Fig. 12.15

BCRP is a parallelogram, so $CR = BP$.
 $BP = PA$ and $\angle AQP = \angle CQR$ (vertically opposite angles).

Combining with $CR \parallel BA$, $\triangle APQ \cong \triangle CRQ$. (Why?)

This gives two equalities.

First, $AQ = QC$, so Q is the midpoint of AC .

Second, $PQ = QR$, so $PQ = \frac{PR}{2} = \frac{BC}{2}$.

We have deduced the following important fact. It can be considered a converse of the Midpoint Theorem, for reasons we will see in Exercise 22 at the end of the chapter.

Theorem 7 (Converse of the Midpoint Theorem): *The line drawn through the midpoint of one side of a triangle and parallel to another side bisects the third side (and the resulting segment has half the length of the parallel side).*

Let us see a different reasoning to justify Theorem 7. Read this short proof carefully.

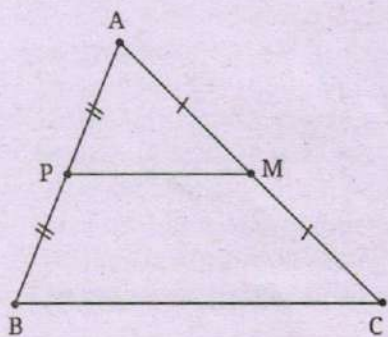


Fig. 12.16

Another proof of Theorem 7. Recall our previous setup: the line parallel to BC through midpoint P of AB meets AC at point Q . Ignore this line for now and instead consider the midpoint M of AC . We want to show $M = Q$. We have $PM \parallel BC$ by the Midpoint Theorem. Therefore line $PM =$ line PQ , because there is a unique line parallel to BC through P . As lines PM and PQ coincide, they must intersect AC in the same point. Thus $M = Q$. This is what we wanted!

We used the Midpoint Theorem to prove its converse! This may look like cheating because, as emphasised earlier, the truth of a statement and of its converse are separate issues. But there is no mistake here. There is a key new point in this proof, namely that there is a unique line through a given point and parallel to a given line. This proof illustrates that sometimes we can use a theorem to prove its converse.

In Grade 10, we will generalise Theorem 7 by dropping the condition that P is the midpoint of AB and assuming only that $PQ \parallel BC$. Can you guess the conclusion in this case? (**Hint:** the name of the more general theorem includes the word proportionality.)

12.3.2 A Surprising Property of Medians

The Midpoint Theorem allows us to deduce an important fact about triangles. Let us discover it experimentally. In $\triangle ABC$, suppose P , Q and R are the midpoints of side AB , AC and BC respectively. Instead of joining the midpoints with each other as we did earlier, draw segments AR , BQ and CP . These are called the **medians** of $\triangle ABC$. What do you see? Repeat this for several triangles and see that the three medians always seem to be concurrent, meaning they pass through a common point!

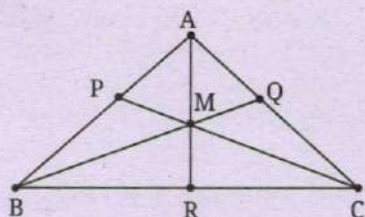


Fig. 12.17

There is more! Assume for the moment that the medians are always concurrent. Let M be the common point where they meet. Compare the lengths of the two parts CM and MP of the median CP , and do the same for the other two medians. Do you see a pattern? It appears that the ratio $CM : MP = 2 : 1$. We will now show why this is always true.

Geometric reasoning: It seems difficult to keep track of the three medians at once, so let us first examine only two medians CP and BQ . Call their intersection point M . We want to show that $CM = 2MP$ and $BM = 2MQ$.

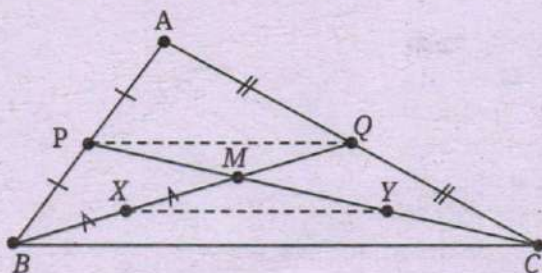


Fig. 12.18

We have not yet seen any theorem that gives a $2:1$ division of a segment. The Midpoint Theorem only gives a $1:1$ ratio. So to show that $CM = 2MP$, one idea is to bisect CM and try to show that the resulting three parts of CP are equal.

Let Y and X be the midpoints of CM and BM respectively. Now we can use the Midpoint Theorem in $\triangle ABC$ and in $\triangle MBC$! We get

that PQ and XY are parallel to BC and hence to each other. Moreover $PQ = XY = \frac{BC}{2}$. So $\triangle MPQ \cong \triangle MYX$ by ASA (How?). Therefore $MQ = MX = XB$ and $MP = MY = YC$. This shows $CM : MP = 2 : 1 = BM : MQ$.

What about the third median AR ? We can repeat the argument in the previous paragraph with medians AR and BQ ! Let N be the point at which the AR and BQ intersect.

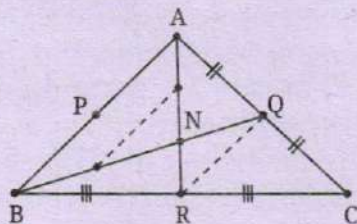


Fig. 12.19

Now the same reasoning gives $BN : NQ = 2 : 1 = AN : NR$. (Use Fig. 12.19 if you wish to repeat the reasoning.) We already had $BM : MQ = 2 : 1$, so $BM : MQ = 2 : 1 = BN : NQ$. But this means M and N have to be the same point, because there is only one point that divides BQ in a given ratio. Thus M lies on all three medians and divides each median in the ratio $2 : 1$. We have deduced the following important fact.

Theorem 8: (Centroid Theorem) *The three medians of $\triangle ABC$ pass through a common point, which divides each of the medians in the ratio $2 : 1$, with the longer part connecting to the vertex. This point is called the centroid of $\triangle ABC$.*

Think and Reflect

Give another proof of the Centroid Theorem using a clever construction that we will describe only partly and ask you to complete. Once again let M be the intersection point of the two medians CP and BQ . We want to show that the intersection point of line AM with side BC is the midpoint of BC . For this, extend AM up to a carefully chosen point S such that $BSCM$ becomes a parallelogram. (**Hint:** To arrange $MC \parallel BS$, first note that PM and MC are along the same line. Now where should M be on segment AS ? Does your placement of S also prove $MB \parallel CS$?)

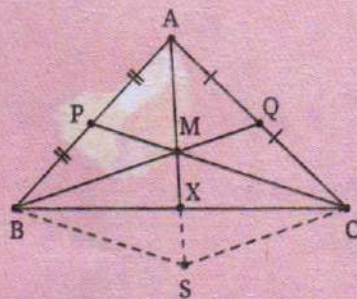


Fig. 12.20

We started this section by examining the triangle formed by midpoints of the sides of a triangle. What happens if we use midpoints of the sides of a quadrilateral to make another quadrilateral? Experiment and see what this new quadrilateral looks like. It appears to be a parallelogram! Let us try to justify this using reasoning.

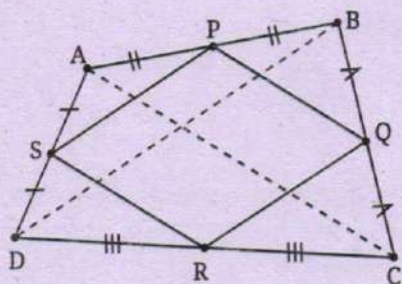


Fig. 12.21

Geometric reasoning. Suppose the midpoints of sides AB , BC , CD and DA of a quadrilateral $ABCD$ are P , Q , R and S respectively. The Midpoint Theorem applied to $\triangle ABC$ and to $\triangle ADC$ gives, respectively, $PQ \parallel AC$ and $SR \parallel AC$. So $PQ \parallel SR$. Using the Midpoint Theorem in $\triangle BCD$ and in $\triangle BAD$ gives $QR \parallel PS$. Therefore $PQRS$ is a parallelogram.

We have deduced the following interesting fact, also known as Varignon's theorem.

Theorem 9 (Midpoint Theorem for Quadrilaterals): *The midpoints of the four sides of a quadrilateral are the vertices of a parallelogram. This is called the Varignon parallelogram of the original quadrilateral.*

EXERCISE SET 12.3

- If P , Q , R are the midpoints of sides AB , AC , BC respectively of $\triangle ABC$, show that $\triangle PQR$ is congruent to $\triangle QPA$ and to two other triangles which you should identify.
 - Suppose someone erases $\triangle ABC$, leaving only $\triangle PQR$ on the paper. Can you reconstruct $\triangle ABC$ from $\triangle PQR$?

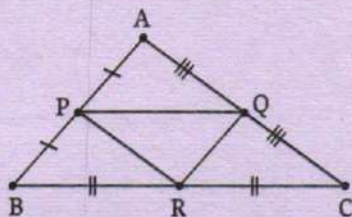


Fig. 12.22

2. In $\triangle ABC$, let M and N be midpoints of AB and AC respectively. Let D be any point on BC . Show that MN bisects AD .

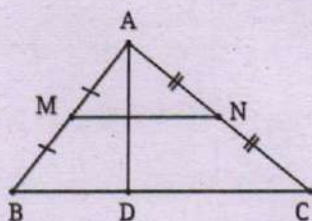


Fig. 12.23

3. In a quadrilateral $ABCD$, suppose $AB \parallel DC$ and $AB \neq CD$. Suppose G and H are the midpoints of AC and BD respectively. Prove that $GH \parallel AB$. (Why did we assume $AB \neq CD$?)

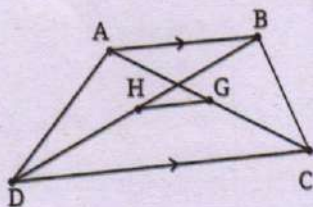


Fig. 12.24

4. Suppose the midpoints of sides AB , BC , CD and DA of a quadrilateral $ABCD$ are P , Q , R and S respectively.
- Show that PR and QS bisect each other.
 - Show that if $AC = BD$, then PR and QS are perpendicular. Is the converse true?
5. Suppose $PQRS$ is the Varignon parallelogram of $ABCD$.

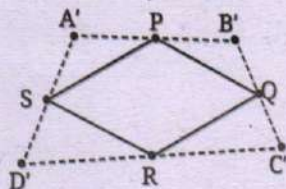
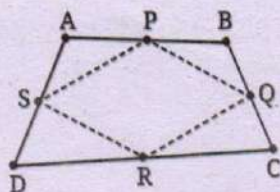


Fig. 12.25

- Copy only $PQRS$ on another paper. Show how you will recreate a congruent copy $A'B'C'D'$ of $ABCD$ from $PQRS$. This will be relevant when we return to study tilings at the end of this chapter. (**Hint:** How will you place vertex A' ? How will you place B' , C' and D' ?)

- (ii) There are multiple ways to construct $A'B'C'D'$ in (i). Justify why the quadrilateral $A'B'C'D'$ you constructed is congruent to $ABCD$. You may need to show why S is collinear with the points A' and D' that you constructed, and similarly for P , Q and R . (**Hint:** Use congruence of triangles, for example $\triangle SDR \cong \triangle SD'R$.)
- (iii) Show that if $PQRS$ is a square, then AC and BD are perpendicular and equal. Prove the converse.

12.4 TILING THE PLANE USING ANY 4-GON

There are many ways to tile the plane, using a single shape or multiple shapes, leading to beautiful pictures. These occur in mosaics in several ancient civilizations to more recent artwork, such as that of M.C. Escher, in nature (for example, a honeycomb) and in decorative patterns we encounter in daily life.

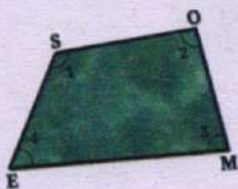
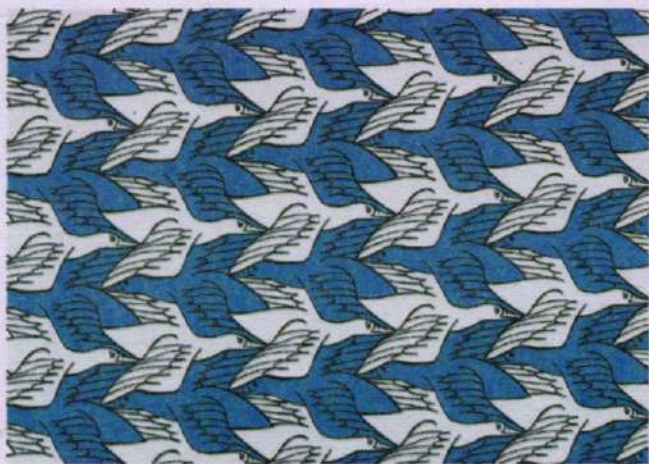


Fig. 12.26

We will see how the plane can be tiled with any given 4-gon, not just a parallelogram as we saw at the beginning of the chapter.

Before you see the answer in Fig. 12.28, see the hints below and try to find a tiling using the given irregular 4-gon $SOME$. First, trace $SOME$ on paper with its four angles labelled 1, 2, 3, 4. Remember that they add up to 360° . Cut out 15 identical copies.

Throughout this section, we will focus on experimental verification of our methods. However, you are encouraged to try to prove, on your own, that they work.

Think and Reflect

Suppose we have a tiling of the plane. Consider any vertex. As we go around this vertex and consider the angles made by consecutive lines, the total of these angles must be 360° . This suggests an idea. What if we take 4 copies of SOME and fit them together around a common point so that each angle is used once as we go around?

There are multiple ways of doing this, as shown in Fig. 12.27. Can you use any of these ways to continue fitting further copies of SOME to tile the plane? Try it with the 15 copies you made!

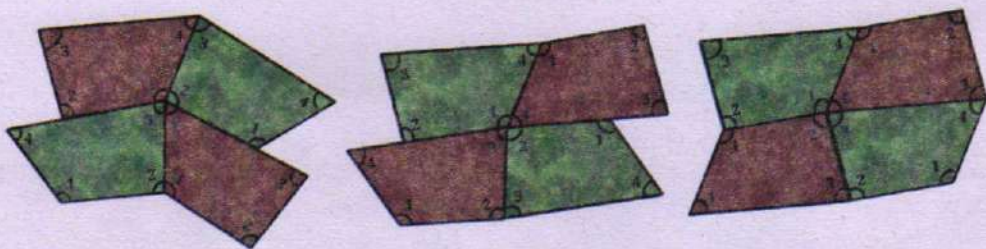


Fig. 12.27

Here is how one can tile! Recreate it with your pieces.

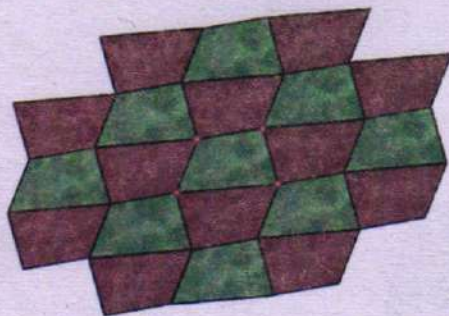


Fig. 12.28

How can we understand the figure? There seems to be a repeating pattern. (1) Can you precisely describe a procedure to draw the pattern

so that someone can draw the tiling on their own, based only on your description? (2) Can you justify why your procedure works?

We will sketch two methods and ask you to complete the details.

Method 1: We start with two copies of SOME, one exactly above the other. Rotate the top copy by 180° around the midpoint of OM. The rotated copy aligns with the original copy perfectly along the shared edge OM.

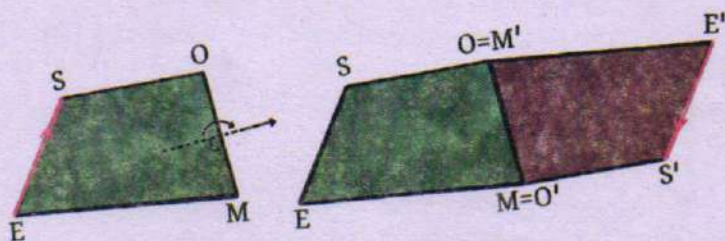


Fig. 12.29

Now repeat this procedure at each edge of every copy of SOME that you add. Verify that this will result in a tiling! Two steps of a possible continuation are shown in Fig. 12.30. The first step (Fig. 12.30 A) adds 4 copies around the starting copy and the second step (Fig. 12.30 B) adds 4 more copies. Note two interesting things about the second step: (1) each new copy can be obtained by rotating any one of its neighbours, and both ways give the same result. (2) The new copies fit perfectly. Can you explain these facts by reasoning? This is needed to prove that the method works!

Do you see which of the three possibilities shown in Fig. 12.27 occurs in the tiling?

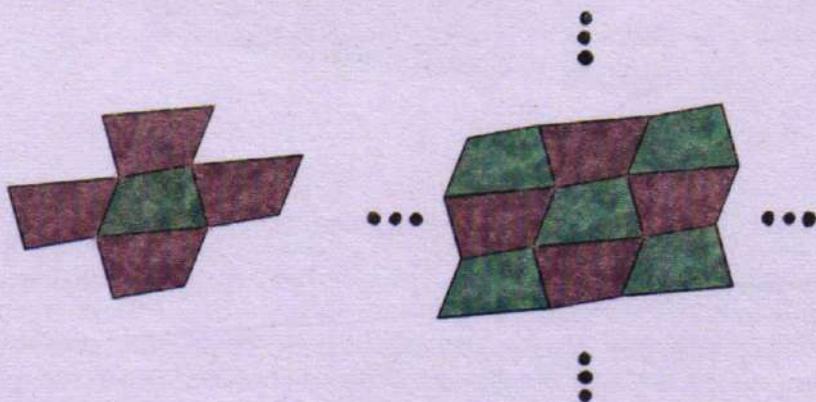


Fig. 12.30 A

Fig. 12.30 B

Method 2. Before we explain Method 2, let us conduct a thought experiment. Remember the Varignon parallelogram for 4-gons (Theorem 9). Imagine that we have drawn this parallelogram in each copy of SOME in the tiling shown earlier in Fig. 12.28. Visualise what the picture looks like. Fig. 12.31 shows the result: It appears that the Varignon parallelograms form a grid, giving the same parallelogram tiling we saw in the introduction!

This suggests an idea. What if we start by drawing just the simple grid tiling made by the Varignon parallelogram of SOME? Can we use this grid to construct the new tiling by SOME?

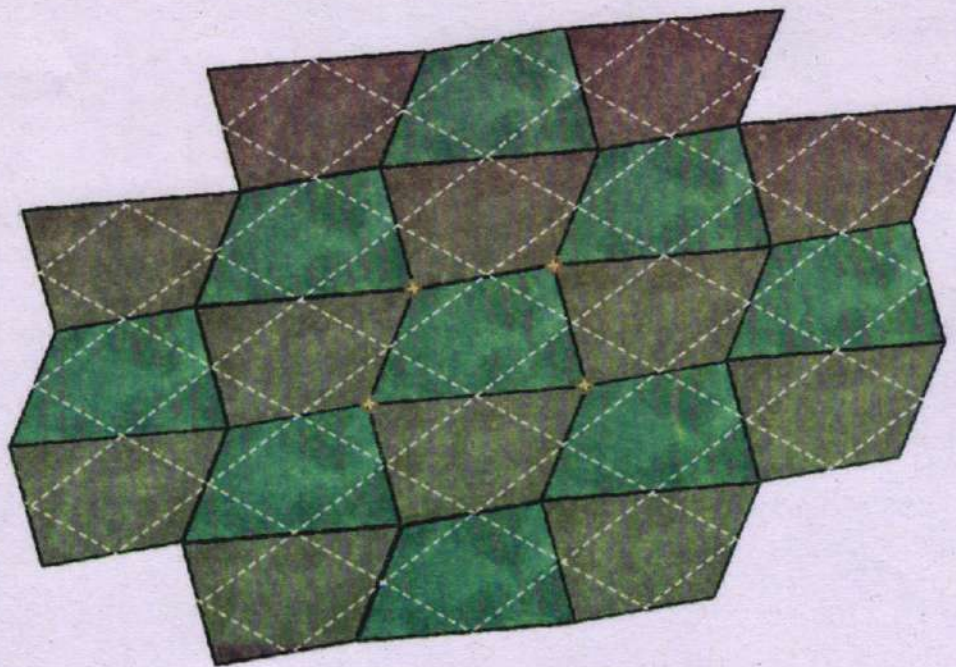


Fig. 12.31

A grid of Varignon parallelograms of SOME is given in Fig. 12.32. To begin with, focus only on the 9 green coloured copies of SOME in Fig. 12.31. We will first place only these 9 copies in Fig. 12.32. The corresponding parallelograms are shaded in Fig. 12.32. Place 9 copies of SOME so as to match the way the green coloured copies of SOME are placed in Fig. 12.31 (You were asked to give a precise description of how to do this in Exercise Set 12.3, Q5.) If done carefully, you will observe that each

4-gon you placed meets other placed copies exactly at vertices. Now see how 4 copies of SOME are made automatically in the gaps! Carefully place four more copies of SOME in these gaps. Continuing this process with more copies of SOME will give us the desired tiling.

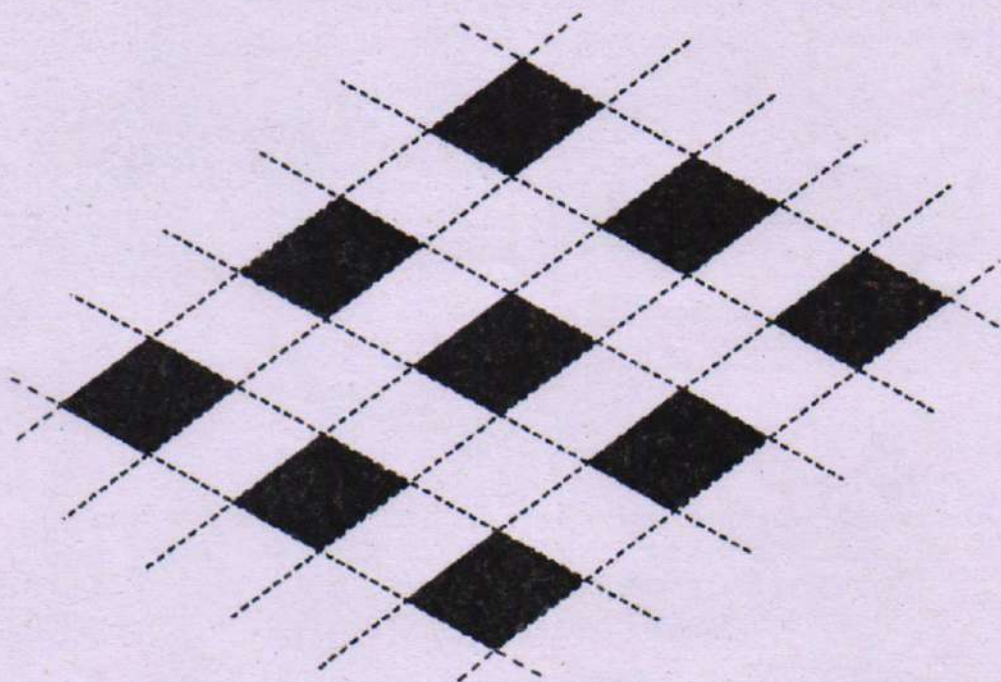


Fig. 12.32

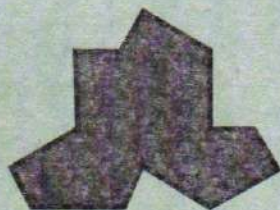
It should be emphasised that our procedure needs to be justified to assure us that it will always work. We will not do this here, but invite you to try it on your own!

Mathematicians have found many interesting ways to tile the plane using a small set of shapes. Some of these, called aperiodic tilings, turned out to be useful to describe atomic arrangements of quasicrystals—new physical structures that were discovered some time later. The discovery of quasicrystals, made by Dan Shechtman, was very surprising because the atomic arrangement was outside what was then considered physically possible. It led to a broader scientific understanding of the nature of order in solids. Dan Shechtman won the chemistry Nobel prize in 2011 for his discovery.

A Cutting-edge Discovery

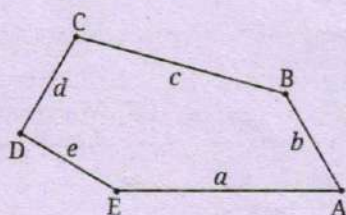
The following shape—the first of its kind!—that leads to aperiodic tilings was discovered only recently in 2023 by a team of four mathematicians (Smith, Myers, Kaplan and Goodman-Strauss).

Cut out 15 identical copies of the following shape (known as the ‘hat’) and start to tile with it. Do you see any pattern? You will find that the hat shape will tile the plane endlessly, but without ever quite repeating the same pattern. That is why it is called an ‘aperiodic’ tiling.



EXERCISE SET 12.4

- Justify why the plane cannot be tiled with a regular pentagon. (**Hint:** Read the first 3 sentences of ‘Think and Reflect’ in the section on tiling.) (There are many ways to tile the plane using a suitable irregular pentagon. The most recent method was found in 2015.)



$$\begin{aligned} A &= 60^\circ \\ B &= 135^\circ \\ C &= 105^\circ \end{aligned}$$

$$\begin{aligned} D &= 90^\circ \\ E &= 150^\circ \end{aligned}$$

$$a = 1$$

$$b = \frac{1}{2}$$

$$c = \frac{1}{\sqrt{2}(\sqrt{3}-1)}$$

$$d = \frac{1}{2}$$

$$e = \frac{1}{2}$$



2. Draw a non-convex 4-gon DART. Show how we can tile the plane with copies of DART. Both methods that we discussed earlier will work. Which do you prefer?

END-OF-CHAPTER EXERCISES

1. Using a fact about parallelograms, show how to tile the plane using any given triangle. (**Hint:** Can you use the parallelogram tiling in the introduction?)
2. Mark the midpoint of the line drawn on the paper (see Fig. 12.33), given that the horizontal lines are equally spaced. Justify your answer.

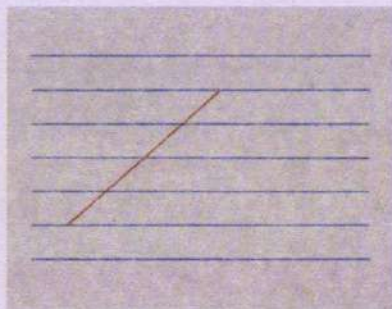


Fig. 12.33

3. You know that the sum of angles of a quadrilateral is 360° , even for a non-convex quadrilateral. (Recall the proof.) Now consider a self-intersecting quadrilateral ABCD, where AB and CD intersect at point E. Show that $\angle A + \angle B + \angle C + \angle D < 360^\circ$. Can you construct ABCD such that $\angle A + \angle B + \angle C + \angle D = 2^\circ$?
4. In a parallelogram ABCD, two points P and Q are taken on diagonal BD such that $DP = BQ$ (see Fig. 12.34). Show that APCQ is a parallelogram.

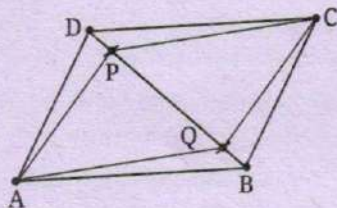


Fig. 12.34

5. A right-triangle shaped cutout of a paper is folded such that point A touches point B (Fig. 12.35). Show that the crease line can be used to find the midpoint of not only AB but also that of AC.

Step 1

Step 2

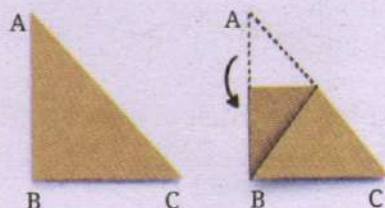


Fig. 12.35

6. You saw how to use the Midpoint Theorem to divide a given triangle into 4 congruent triangles. Can we divide a triangle into 3 congruent triangles? This exercise shows us how to do that and more, provided we are allowed to cut and reassemble.
- Draw medians AP, BQ and CR of $\triangle ABC$, meeting in a common point M. Cut along each median to get 6 triangles. Show with justification how to assemble the 6 pieces into 3 congruent triangles! (**Hint:** Align $\triangle MPB$ and $\triangle MPC$ along equal sides PB and PC, matching P with itself and B with C.) Prove that you get a triangle. What are its side lengths?
 - Repeat the procedure with each of the 3 assembled triangles you got in (i). Show that each of the resulting 9 triangles has sides $\frac{AB}{3}$, $\frac{BC}{3}$ and $\frac{AC}{3}$.

The procedure and the result above were unnoticed till 2014, when they were discovered by Lee Sallows, an amateur mathematician!

- Naturally, we can next ask about the possibility of cutting a triangle and reassembling it into 2 congruent triangles. You were asked this question in the chapter *Measuring Space: Perimeter and Area*. Can you solve this now? (**Hint:** Use median AP in $\triangle ABC$ and then a median of $\triangle APB$.)

For any two given figures with straight sides, we can ask whether one figure can be cut (using only straight cuts) into pieces and reassembled to make the other. This notion is

called scissors congruence. The figures need not be in the plane. Research on such questions at a higher level is ongoing.

7. **A more general midpoint theorem and its converse.** In a quadrilateral $ABCD$, suppose $AB \parallel DC$. Recall that such $ABCD$ is called a trapezium. Let E be the midpoint of AD . A line drawn through E intersects side BC at F .

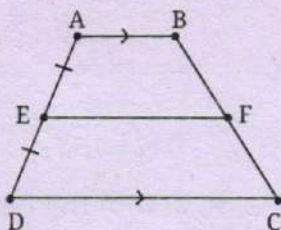


Fig. 12.36

- (i) If $EF \parallel AB$, then show that F is the midpoint of BC . Conclude that $EF = \frac{(AB + CD)}{2}$.
- (ii) If F is the midpoint of BC , then show that $EF \parallel AB$.

There are at least two ways to solve (ii). Do it directly by using the midpoint M of BD and showing that EM and FM are the same lines. Or use (i) along with the same idea used in the second proof of the converse of the Midpoint Theorem. Which way do you think is simpler?

8. The diagonals AC and BD of a parallelogram $ABCD$ intersect at O . A line through O meets AB and CD at points P and Q respectively. Show that O is the midpoint of PQ . (Multiple proofs are possible. Which is the simplest?)

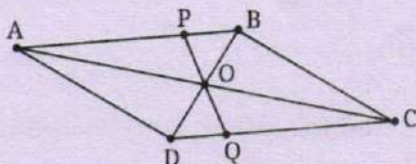


Fig. 12.37

- *9. $ABCD$ is a trapezium with parallel sides $AD = 3$ cm and $BC = 5$ cm. E and F are the midpoints of the non-parallel sides. Find the ratio of the areas of the 4-gons $AEFD$ and $EBCF$.

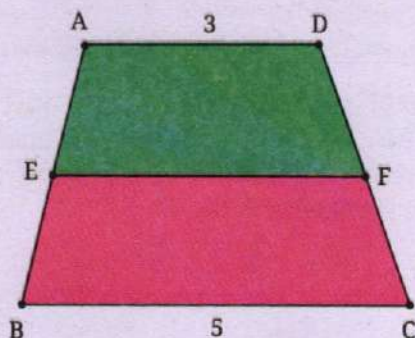


Fig. 12.38

10. Consider 4 points A, B, C, D in the plane with no three collinear. Answer the following questions. Some answers may require you to consider different cases, depending on how the points are positioned in the plane.
- How many different quadrilaterals do they form if the quadrilateral is allowed to be self-intersecting or non-convex?
 - How many of these quadrilaterals are self-intersecting? How many are convex?
11. Suppose P is a point on side AB of $\triangle ABC$ and the line through P parallel to BC meets AC in point Q. For parts (i) to (iii), assume $AP = 1$, $AQ = \sqrt{5}$, and see Fig. 12.39.

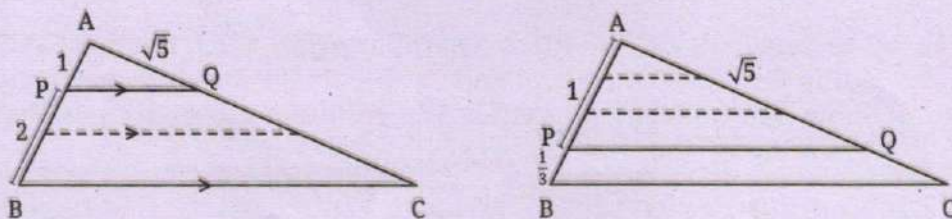


Fig. 12.39

- If $PB = 2$ find QC . (**Hint:** See the dotted line)
- If $PB = \frac{1}{3}$ find QC . (**Hint:** See the dotted lines)
- If $PB = \frac{2}{3}$ find QC .
(**Hint:** Divide both PB and AP suitably.)
- Show that if $\frac{AP}{PB}$ is a rational number, then $\frac{AP}{PB} = \frac{AQ}{QC}$. In Grade 10, you will prove this equality without assuming $\frac{AP}{PB}$ is rational. That will require a new idea.

12. (i) Suppose $ABCD$ is a parallelogram and M, N are midpoints of AB and CD respectively. Show that segments DM and BN trisect segment AC .
- (ii) Use part (i) to find a procedure to trisect any given segment PQ . Find two other ways to trisect PQ , one using Exercise 11 above and a third using the Centroid Theorem.

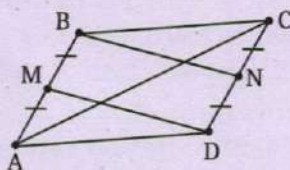


Fig. 12.40

13. Is the Midpoint Theorem for Quadrilaterals (Theorem 9) true when the quadrilateral is non-convex? How about when it is self-intersecting? Experiment and check. The pictures may look strange, but the theorem appears to be true.
- (i) Can you explain why this is so? See the geometric reasoning in the text.
- (ii) There is one exception: In a very special case the Varignon parallelogram becomes a single segment. When will this happen?
- *(iii) Does your reasoning apply even when $ABCD$ is non-planar?
14. Let P, Q, R, S be four points on sides AB, BC, CD and DA respectively of a quadrilateral $ABCD$. Suppose $PQRS$ is a parallelogram. Must P, Q, R and S be midpoints of the respective sides? This can be considered a possible converse question to Theorem 9.
15. (i) Complete the following proof of the Midpoint Theorem. Extend segment PQ beyond Q until point S . By how much should we extend PQ ?

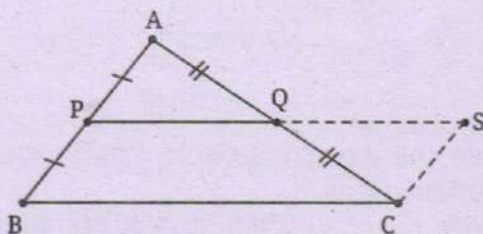


Fig. 12.41

It would be good to be able to prove that $\triangle APQ \cong \triangle CSQ$. (Why?) Use this as a guide to specify the location of S and then complete this proof.

- (ii) Give a similar proof of the converse of the Midpoint Theorem (Theorem 7). Start by extending PQ up to a suitable point S.
- *16. Review all the properties of a rhombus/rectangle/square that you proved in Grade 8. Formulate a converse of each. Decide if the converse is true. There are many possibilities here!
- *17. Show that the sum of angles of a non-planar quadrilateral is always less than 360° . Can you find a non-planar quadrilateral ABCD for which $\angle A + \angle B + \angle C + \angle D = 2^\circ$?
(Hint: Think of a diagonal, say AC, as a hinge around which triangles ABC and ADC can rotate.) What happens to each angle of ABCD as you do this rotation?
- *18. Let us see a third method to tile the plane using a 4-gon.

Focus on only two coloured copies of SOME sharing a vertex. What do you see? It appears that each 4-gon is just a shifted copy of the other. Let us see exactly how. Draw two copies of SOME as shown in Fig. 12.42 so that the diagonals EO and E'O' are collinear with O = E'. Now place a cutout of one copy on top of SOME with diagonal EO drawn on it. Slide this cutout so that segment EO moves along EO' until EO matches E'O'. Verify that your cutout exactly matches S'O'M'E'.

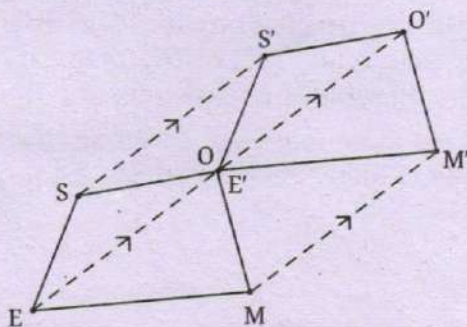


Fig. 12.42

- (i) Prove the exact match of SOME with S'O'M'E' using four parallelograms.
- (ii) Follow the described procedure along each diagonal of the starting coloured 4-gon in both directions to get 4 new 4-gons. Using these 4 copies, repeat the procedure 4 more

times to place 4 new copies, resulting in a 3 by 3 grid of 9 copies exactly like the 9 coloured copies in Fig. 12.42. These 9 copies meet each other only at vertices and once again the blank spaces among them also form 4-gons congruent to SOME!

***19. Is there a 4-gon with given side lengths?**

- (i) Recall the following fact about triangles and check it by construction. For given positive numbers a, b, c , is there a triangle whose sides have these lengths? The answer is Yes exactly when the sum of any two numbers is greater than the third. If we arrange the numbers in increasing order (suppose $a \leq b \leq c$), then this amounts to requiring $a + b > c$. (**Hint:** Start by drawing a segment of length c .)
- (ii) Suppose a 4-gon has 2, 5, 11 as three side lengths. Can the length of the fourth side be 100? Can it be 10? Can it be 1? What are the possible lengths of the fourth side?
- (iii) For given positive numbers a, b, c, d , how will you decide if there is a 4-gon whose sides have these lengths?

***20. Counting diagonals of a polygon.**

- (i) How should we define a diagonal of an n -gon? How many diagonals does an n -gon have? Make a table for small values of n . A 3-gon has no diagonals. A 4-gon has 2. How many diagonals does a 5-gon have? A 6-gon? Try to find a pattern and guess the answers for $n = 7$ and $n = 8$. Check your guesses by systematic counting.
- (ii) Can you guess a formula for the number of diagonals? How many diagonals get added when we increase the number of sides by 1? Can you now justify why the formula you guessed is true for all n ?

***21. Sum of angles of a polygon.** What is the sum of angles of a (planar non-self-intersecting) n -gon? We know that the answer is 180° for $n = 3$ and 360° for $n = 4$. Find the next few values. Then find a formula in terms of n and prove it.

***22. Multiple converses to a theorem.** Let us see how multiple statements can be considered converses to the Midpoint Theorem and how Theorem 7 is one of them.

To formulate a converse we should express the original statement in "If ... then ..." form. For a complex statement, there may be multiple ways to do that. The Midpoint Theorem starts with $\triangle ABC$ and points P and Q on sides AB and AC respectively.

The theorem has two assumptions and two conclusions, which we have named for further discussion.

Assumptions: (P MID) P is the midpoint of AB, and (Q MID) Q is the midpoint of AC.

Conclusions: (PRL) $PQ \parallel BC$, and (HALF) $PQ = \frac{BC}{2}$.

Let us use these four named conditions to discuss various possible statements.

- (i) A natural converse of the Midpoint Theorem would be: "If (PRL) and (HALF) are true, then (P MID) and (Q MID) are true." Write out this statement fully. Experiment and see that it seems to be true! Prove the statement. (**Hint:** Try to run a proof of the Midpoint Theorem backwards.)
- (ii) To see Theorem 7 as a converse of the Midpoint Theorem, we first write the Midpoint Theorem as follows: "Suppose P is the midpoint of side AB of $\triangle ABC$ and Q is a point on side AC. If Q is the midpoint of AC then $PQ \parallel BC$." (Also $PQ = \frac{BC}{2}$, but let us set that aside for now.) Write the converse of the second sentence in quotes by keeping the first sentence the same. Verify that Theorem 7 is what you get! (And $PQ = \frac{BC}{2}$ also follows.) Now write Theorem 7 in terms of the four named conditions.
- (iii) The discussion so far suggests that it is reasonable to take two of the four listed statements as assumptions and ask if the other two are true. Verify that only one such combination remains to be examined. "If (P MID) and (HALF) are true, then can we conclude (Q MID) and/or (PRL)?" Write this out in words. This is a precise version of question (1) stated before Theorem 7. Can you answer it?

In the text, why did we focus only on Theorem 7 among the multiple possibilities? Because Theorem 7 turns out to be particularly useful. It is still important to pursue various avenues while exploring a subject: often it is only after examining multiple possibilities that we can judge what is most valuable.

- *23. **Validity of tiling methods.** Show using reasoning that each of the three tiling methods we saw produces a tiling of the plane

using the given 4-gon. You have to prove that when we place new copies using any of the three procedures, the entire plane is covered with no gaps and no overlaps. First, think carefully about what it means to prove this. Then find a proof for each procedure.

- *24. What fraction of the square is shaded?

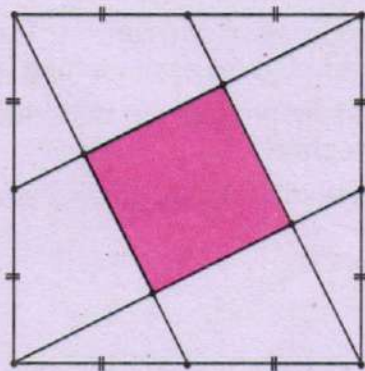


Fig. 12.43

CHAPTER SUMMARY

- **Quadrilateral:** For four distinct points A, B, C, D in a plane with no three collinear, the **quadrilateral** ABCD is a collection of all points on segments AB, BC, CD, DA provided every point other than A, B, C, and D lies on exactly one of these four segments.
- A quadrilateral is **convex** when all of its internal angles are less than 180° , or equivalently, when its diagonals intersect.
- The following converse statements about parallelograms are true:
 - If the opposite sides of a 4-gon are equal, then it is a parallelogram.
 - If the opposite angles of a 4-gon are equal, then it is a parallelogram.
 - If the diagonals of a 4-gon bisect each other, then it is a parallelogram.
 - If a pair of opposite sides of a 4-gon are equal and parallel, then it is a parallelogram.

- **The Midpoint Theorem:** the line joining the two midpoints of any two sides of a triangle is parallel to the third side and has half the length of the third side.
- **A Converse to the Midpoint Theorem:** the line passing through the midpoint of a side of a triangle and parallel to a second side bisects the third side.
- Medians of a triangle are concurrent and the point of concurrence, called the **centroid**, divides each median in a 2 : 1 ratio.
- The 4-gon formed by joining the midpoints of a given 4-gon is always a parallelogram.
- The plane can be tiled with any given 4-gon.



In Chapter 2 of *Ganita Manjari*, Part I, we explored linear patterns and learnt about linear polynomials. In this chapter, we will go a step further in exploring linear equations in two variables, by learning how to solve a **pair** of linear equations.

13.1 LINEAR EQUATIONS IN TWO VARIABLES

Radha stops at a fruit cart. The price tags say that mangoes cost ₹60 per kg and bananas cost ₹50 per kg. She buys some kilograms of mangoes and some kilograms of bananas. Altogether, she pays ₹280. Can we express this situation as an equation?

Since we do not know how many kilograms of mangoes or bananas she bought, let us assume that

x = number of kilograms of mangoes

y = number of kilograms of bananas

The cost of the mangoes is ₹60 x and the cost of the bananas is ₹50 y .

The total cost is ₹280. Hence, we can express this using the linear equation

$$60x + 50y = 280.$$

Notice that the left-hand side (LHS) and the right-hand side (RHS) of the equation both represent the same quantity, that is, the total cost (in ₹).

The equation $60x + 50y = 280$ is a **linear equation in two variables**.



13.1.1 Standard Form of Linear Equations in Two Variables

Any equation that can be expressed in the form $ax + by + c = 0$, where a , b and c are real numbers, and a and b are not both zero, is called a **linear equation** in two variables.

$ax + by + c = 0$ is referred to as the **standard form** of the linear equation. Here a and b are the coefficients of the variables x and y respectively and c is a constant.

It is useful to note the following:

- (i) The coefficients a and b and the constant c can also be rational numbers. For example, $\frac{1}{4}x + \frac{3}{5}y - \frac{1}{2} = 0$ is also a linear equation in two variables in standard form. However, for convenience, in such cases we often multiply the equation by a suitable number to remove the fractions and write the equation with integer coefficients. Here we can multiply by 20 to get the equation $5x + 12y - 10 = 0$.
- (ii) Equations of the type $ax + c = 0$ are special cases of linear equations in two variables as they can be expressed as $ax + 0 \times y + c = 0$. For example, $7 + 4x = 0$ can be written as $4x + 0 \times y + 7 = 0$, with $a = 4$, $b = 0$ and $c = 7$.

Example 1: Write each of the following equations in standard form, $ax + by + c = 0$, and indicate the values of a , b and c in each case:

- (i) $5x + 2y = 2.7$
- (ii) $\frac{x}{2} - 4 = \frac{3}{2}y$
- (iii) $4 = 1.5x - 1.3y$
- (iv) $3y = 5$

Solution:

- (i) $5x + 2y = 2.7$ can be written as $5x + 2y - 2.7 = 0$. Here $a = 5$, $b = 2$ and $c = -2.7$. The equation can also be rewritten as $-5x - 2y + 2.7 = 0$. In this case $a = -5$, $b = -2$, $c = 2.7$.
- (ii) $\frac{x}{2} - 4 = \frac{3}{2}y$ can be written as $\frac{x}{2} - \frac{3}{2}y - 4 = 0$.
Here $a = \frac{1}{2}$, $b = -\frac{3}{2}$ and $c = -4$.
- (iii) $4 = 1.5x - 1.3y$ can be written as $1.5x - 1.3y - 4 = 0$.
Here $a = 1.5$, $b = -1.3$ and $c = -4$. It can also be written as $-1.5x + 1.3y + 4 = 0$. In this case $a = -1.5$, $b = 1.3$ and $c = 4$.
- (iv) $3y = 5$ can be written as $0 \times x + 3y - 5 = 0$. Here $a = 0$, $b = 3$ and $c = -5$.

EXERCISE SET 13.1

1. Write a linear equation in two variables in which $a = 3$, $b = 0$ and $c = -\frac{1}{5}$.
2. Complete the following table after expressing the given linear equations in standard form.

Linear equation in two variables	Standard Form	Coefficient of x	Coefficient of y	Constant term
$y - 15 = \sqrt{2}x$				
$3y - 2x = 0$				
$5x = 3y$				
$x = 8$				
$3y = 1$				

3.
 - (i) The cost of a notebook is twice the cost of a pen. Consider the cost of a notebook to be ₹ t and that of a pen to be ₹ p . Charlie wrote $t = 2p$, whereas Meera wrote $p = 2t$. Which of these two representations is correct?
 - (ii) In a one-day International Cricket match between India and Sri Lanka played in Nagpur, two Indian batsmen together scored 176 runs. Manisha expressed this situation as $x + y = 176$, where the number of runs scored by one batsman is x , and the number of runs scored by the other is y . Is this a correct representation?

13.2 SOLUTION OF LINEAR EQUATION IN TWO VARIABLES

In earlier grades, we have seen how to solve linear equations in one variable. What can we say about the solution of a linear equation involving two variables?

13.2.1 Algebraic Meaning of Solution

A **solution** of a linear equation in two variables is a pair of values of x and y which satisfy the given equation.

Let us consider the equation $3x + 2y = 12$.

Here, $x = 2$ and $y = 3$ is a solution because substituting the values of x and y into the given equation satisfies the equation.

Note that, $3x + 2y = (3 \times 2) + (2 \times 3) = 12$.

A solution can be written compactly as a pair of numbers. For example, the solution $x = 2$ and $y = 3$ can be written as $(2, 3)$. The order of the numbers in this pair matters: the first number is the x -value, and the second number is the y -value. Hence we call this an **ordered pair**.

Similarly, $(4, 0)$ is also a solution of the equation $3x + 2y = 12$ since substituting $x = 4$ and $y = 0$ leads to $3x + 2y = (3 \times 4) + (2 \times 0) = 12$.

What about the ordered pair $(1, 3)$?

We see that $3x + 2y = 3 \times 1 + 2 \times 3 = 9 \neq 12$. So, $(1, 3)$ is not a solution.

Think and Reflect

Are $(1, \frac{9}{2})$, $(0, 6)$, $(4, 1)$ and $(\frac{1}{2}, \frac{21}{4})$ solutions of the equation $3x + 2y = 12$?

Can you find any other solution? How many solutions can you find?

Suppose we are given a linear equation in two variables. For any value u that we assign to x , we can find a value v for y such that (u, v) is a solution. We first substitute u for x in the equation. This gives us a new linear equation in y that can be solved to find v . This allows us to generate an infinite number of solutions to the original equation.

13.2.2 Geometric Visualisation of Solutions

Consider all the solutions of a linear equation in two variables, expressed as ordered pairs. What do we get when we plot the points corresponding to these ordered pairs?

The equation $3x + 2y = 12$ can be rewritten as

$$2y = -3x + 12 \text{ or } y = -\frac{3}{2}x + \frac{12}{2} \text{ which is the same as } y = -\frac{3}{2}x + 6$$

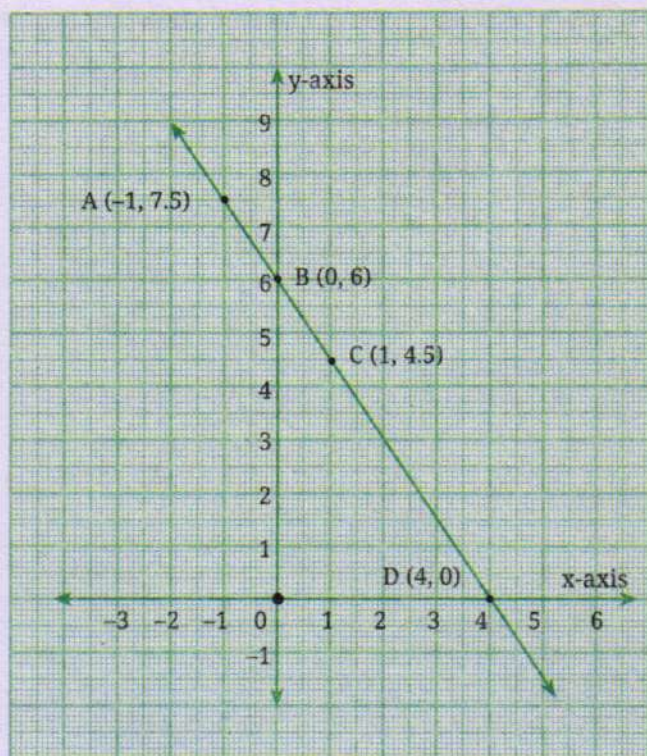


Fig. 13.1

We have already seen in the chapter on Linear Polynomials that plotting the solutions of an equation of the form $y = mx + d$ leads to a straight line joining them. Every solution (x, y) of the equation is a point on the line.

Clearly the solutions of $y = -\frac{3}{2}x + 6$ are the same as those of $3x + 2y = 12$. Hence the graph of $3x + 2y = 12$ is the same as that of $y = -\frac{3}{2}x + 6$.

Example 2: Find any two solutions of $2x + y = 7$. Draw the graph of this equation. Use the graph to find 4 more solutions.

When we substitute $x = 0$ in the equation, we get $(2 \times 0) + y = 7$, that is, $y = 7$. Hence one solution is $(0, 7)$. By substituting $y = 0$, we get $2x + 0 = 7$, that is, $x = \frac{7}{2}$. Thus, $(\frac{7}{2}, 0)$ is also a solution.

We can represent these solutions or ordered pairs in the form of a table.

x	0	$\frac{7}{2}$
y	7	0

Using these two points, we can draw the graph of the equation.

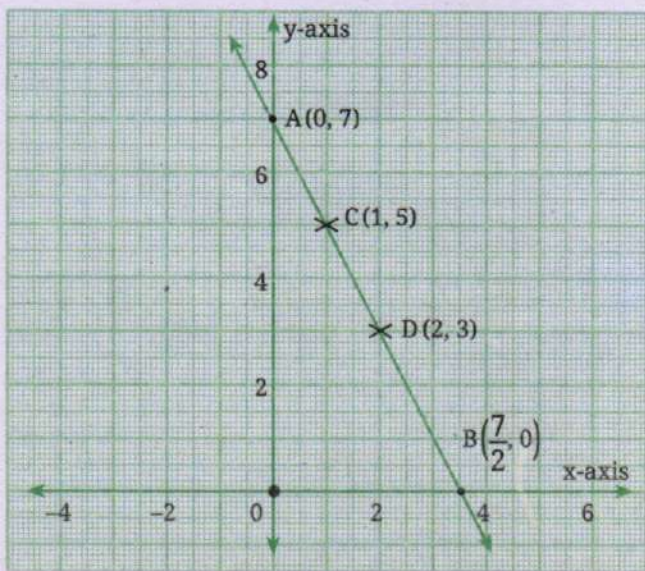


Fig. 13.2: Graph of the linear equation $2x + y = 7$

From the graph, we see that $(1, 5)$, $(2, 3)$, $(3, 1)$, $(\frac{3}{2}, 4)$ are also solutions of $2x + y = 7$.

Note that an easy way of identifying two points on a line is to substitute $x = 0$ in the linear equation and obtain the corresponding value of y . This will give us the point where the line intersects the y -axis. Similarly, we can substitute $y = 0$ and obtain the corresponding value of x . This will give us the point where the line intersects the x -axis.

Example 3: Find any two solutions for each of the following equations and draw their graphs:

(i) $4x + 3y = 12$

(ii) $2x + 5y = 0$

Solution:

(i) Substituting $x = 0$ in the equation, we get $3y = 12$, that is, $y = 4$. So $(0, 4)$ is a solution of the given equation. Similarly, by substituting $y = 0$, we get $x = 3$. Thus, $(3, 0)$ is also a solution.

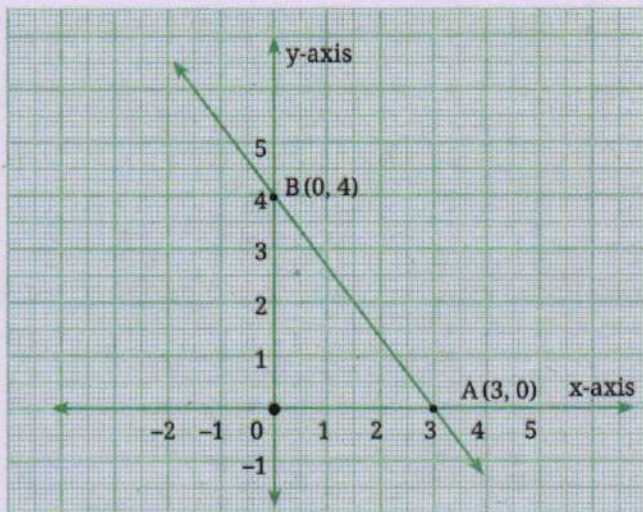


Fig. 13.3: Graph of the linear equation $4x + 3y = 12$

(ii) Substituting $x = 0$, we get $5y = 0$, that is, $y = 0$. So $(0, 0)$ is a solution of the given equation. To get another solution, substitute $x = 1$. The corresponding value of y is $-\frac{2}{5}$. So, $(1, -\frac{2}{5})$ is also a solution. In fact, we can substitute any value for x and obtain a corresponding value of y . For example, substituting $x = 5$ we obtain $y = -2$. Let us represent these solutions or ordered pairs in the form of a table.

x	0	1	5
y	0	$-\frac{2}{5}$	-2

When we plot the ordered pairs A(0, 0), B($1, -\frac{2}{5}$) and C(5, -2) we see that they lie on a line.

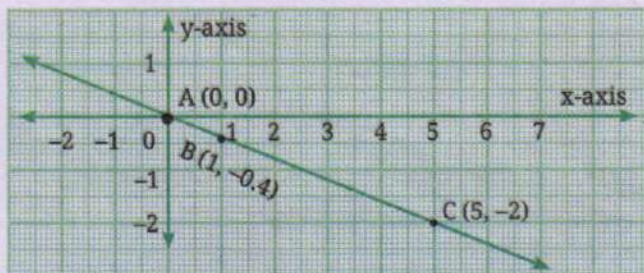


Fig. 13.4: Graph of the linear equation $2x + 5y = 0$

Example 4: In the equations shown below, a and b are unknown numbers.

$$\begin{aligned} 3ax + 4y &= -2 \\ 2x + by &= 14. \end{aligned}$$

If $(-3, 4)$ is the solution of both equations, find the values of a and b .

Solution:

Step 1: Substitute $(-3, 4)$ into the first equation, (that is, $x = -3$ and $y = 4$).

$$\begin{aligned} 3a(-3) + 4(4) &= -2 \\ \text{or} \quad -9a + 16 &= -2 \\ \text{or} \quad -9a &= -18 \\ \text{or} \quad a &= 2 \end{aligned}$$

Step 2: Substitute $(-3, 4)$ into the second equation.

$$\begin{aligned} 2(-3) + b(4) &= 14 \\ \text{or} \quad -6 + 4b &= 14 \\ \text{or} \quad 4b &= 20 \\ \text{or} \quad b &= 5 \end{aligned}$$

Think and Reflect

Four friends—Ranju, Meena, Farhan, and Toshi—are solving problems.

Ranju: I noticed something! If $c = 0$ in the standard form of a line $ax + by + c = 0$, then the line must pass through the origin. Look, if I substitute $x = 0$ and $y = 0$, in equation $ax + by = 0$, the equation is satisfied. So, the origin lies on the line! We can also say that the line passes through the origin.

Farhan: Let us try for the equation $2x + 3y = 0$. Here $a = 2$, $b = 3$ but $c = 0$. If we substitute $x = 0$, then we get $3y = 0$ or $y = 0$. This means $(0, 0)$ lies on the line. So yes, this line passes through the origin.

Toshi: Suppose our equation has $b = c = 0$, say, $5x = 0$. This becomes $x = 0$ which is the equation of the y -axis. And the y -axis passes through the origin.

Meena: And if we take $a = c = 0$, say, $7y = 0$, that means $y = 0$, which is the equation of the x -axis that also passes through the origin.

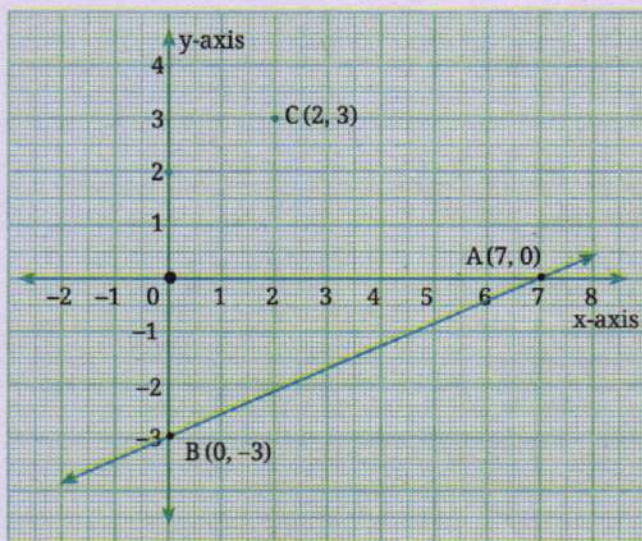
So, they conclude: Whenever $c = 0$, the line $ax + by + c = 0$ will always pass through the origin, irrespective of the values of a or b . Do you agree with them?

EXERCISE SET 13.2

1. Verify if the ordered pair $(4, 3)$ is a solution of $5x - 6y = 2$. Explain your reasoning.
2. Find any two solutions for each of the following equations:
 - (i) $7x - 3y = 21$
 - (ii) $2x + 3y = 5$
3. In the equations given below, m and n are unknown constants: $2mx + 3y = 7$; $4x + ny = -10$. If $(2, -1)$ is the solution of both equations, find the values of m and n .
4. Find two solutions which lie in different quadrants for each of the following linear equations. Identify the quadrants in which the points lie.
 - (i) $5x + 3y = 7$
 - (ii) $5x - 3y = 7$
 - (iii) $-5x + 3y = 7$
 - (iv) $-5x - 3y = 7$

Verify your solutions by representing the linear equations on a graph paper.

5. Consider the graph of the equation $3x - 7y = 21$ shown below. Does the point $C(2, 3)$ lie on the line? Does it satisfy the equation? Can points that do not lie on the line satisfy the equation?



6. State whether the following sentences are True or False. Justify your answer.
- (i) A linear equation in two variables has only one solution.
 - (ii) The graph of a linear equation in two variables always passes through the origin.
 - (iii) A linear equation in two variables can never have rational solutions.
 - (iv) $x = 3$ is a valid linear equation in two variables.
 - (v) The equation $2x + 3y = 7$ has infinitely many solutions.
 - (vi) The point $(1, 2)$ is a solution of the equation $2x + 3y = 7$.
7. (i) Compare the solutions of the equations $3x + 4y = 7$ and $6x + 8y = 14$. Argue that they have the same set of solutions, that is, every solution of one is also a solution of the other.
- (ii) Show that the equations $ax + by = c$ and $kax + kby = kc$, with $k \neq 0$, have the same set of solutions.

13.3 SLOPE-INTERCEPT FORM OF A LINEAR EQUATION IN TWO VARIABLES

13.3.1 Understanding Slope

Have you ever walked or cycled up a steep hill? How did it feel compared to walking on a flat road? On a flat road, you can move without much effort. But as the road becomes steeper, you have to work harder to climb.

The steeper it is, the more effort is needed to cycle.

In mathematics, the idea of 'steepness' of a line is called its **slope** or **gradient**.

1. A **positive slope** means going 'upward' as we move from left to right.
2. A **negative slope** means going 'downward' as we move from left to right.

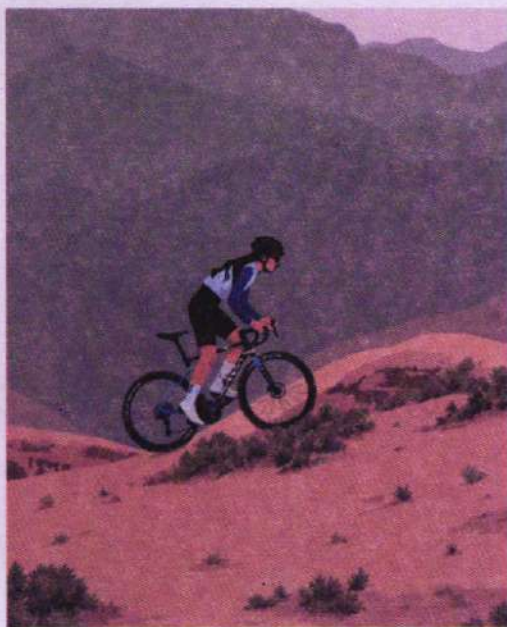


Fig. 13.5: (i) represents the case of a positive slope, whereas (ii) represents that of a negative slope.

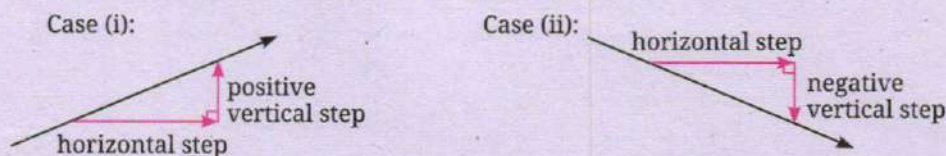


Fig. 13.5

Let us consider a line passing through two points $A(x_1, y_1)$ and $B(x_2, y_2)$.

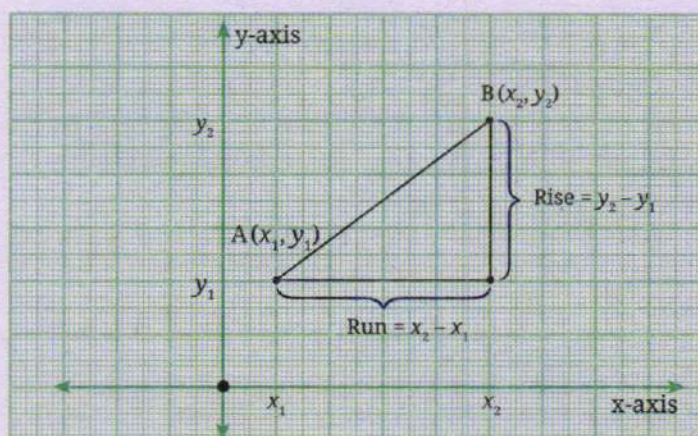


Fig. 13.6

The horizontal distance between the two points is $(x_2 - x_1)$ and is often referred to as the **Run**. It tells you how far the line goes to the left or right. Moving to the right means a positive run while moving to the left means a negative run.

The vertical distance between the two points is $(y_2 - y_1)$ and is often referred to as the **Rise**. It tells you how much the line goes up or down when moving from one point to another. If the line goes up, the rise is positive. If the line goes down, the rise is negative.

$$\text{The slope of a line} = \frac{\text{Rise}}{\text{Run}} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Consider the graph of the equation shown in Fig. 13.7 which passes through the points $A(2, 0)$ and $B(4, 3)$.

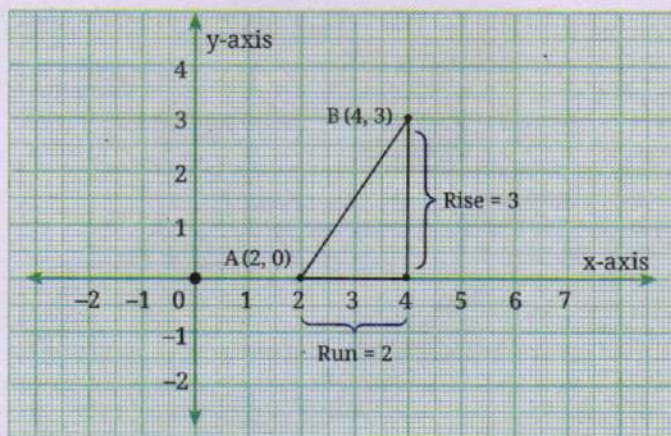


Fig. 13.7

Point B can be reached from point A by moving 2 units along the positive x-direction (the run), and 3 units along the positive y-direction (the rise). Hence slope of the line = $\frac{\text{Rise}}{\text{Run}} = \frac{3}{2}$.

Now consider the same line AB further extended beyond B as shown in Fig. 13.8.

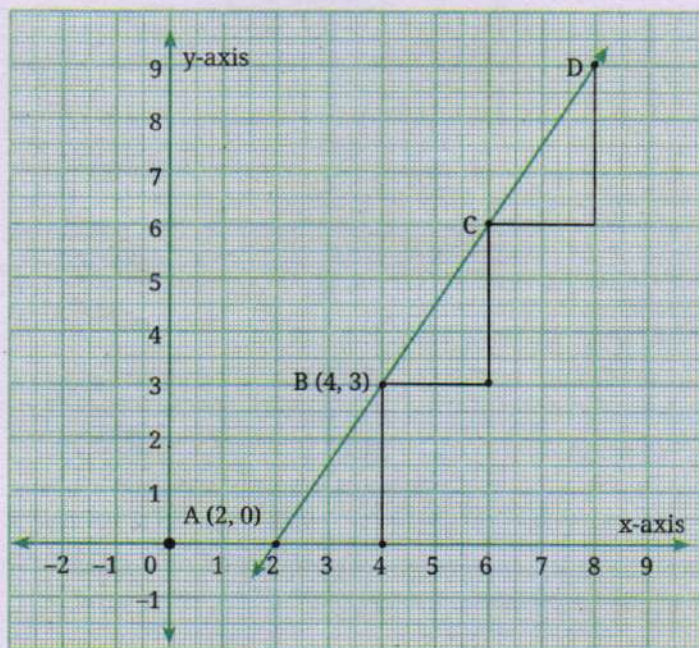


Fig. 13.8

The slope of the line using points A and B is $\frac{3}{2}$ and using B and D we get $\frac{6}{4} = \frac{3}{2}$. We may therefore conclude that the slope of the line remains the same and is independent of where we measure it on the line.

Think and Reflect

1. Consider the point on the line AB whose x-coordinate is 8. What is the y-coordinate of this point? Express this as an ordered pair.
2. What is the x-coordinate of the point on the line AB whose y-coordinate is 6? Express this as an ordered pair.

Example 5: Consider the straight line that passes through A(-1, 0), B(1, 2), and C(4, 5). Plot the points and find the slope of the line.

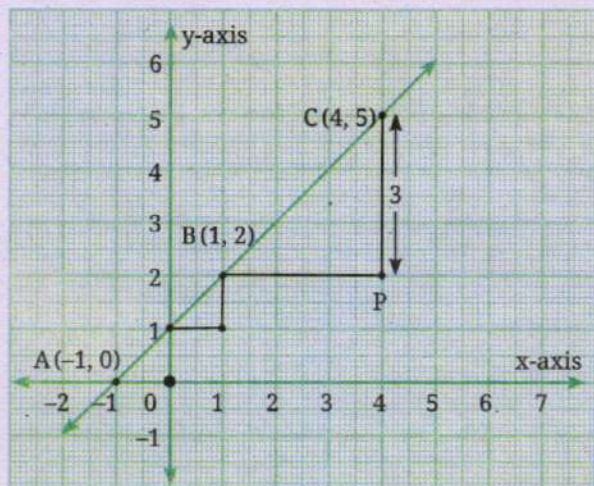


Fig. 13.9

Take any two points, say B and C, and find the ratio of vertical step (rise) to the horizontal step (run).

$$\text{Hence, slope} = \frac{CP}{BP} = \frac{5-2}{4-1} = 1.$$

It may be verified that we will get the same slope by considering any two points.

Example 6: Consider the line in Fig. 13.10 which passes through A(-4, 5) and B(-1, 2). Find the slope of the line.

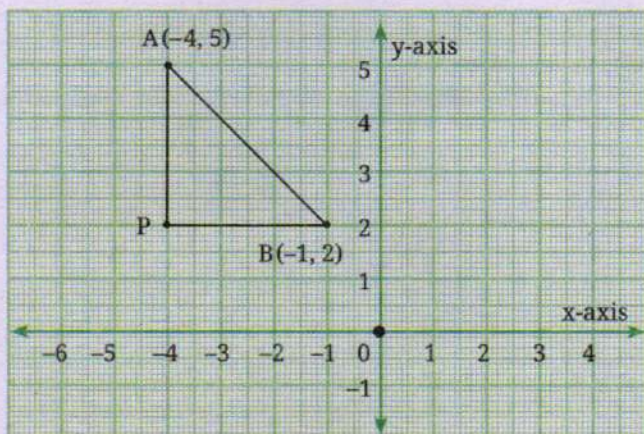


Fig. 13.10

If we denote the points $A(-4, 5)$ and $B(-1, 2)$ as (x_1, y_1) and (x_2, y_2) respectively, then

$$\text{Slope} = \frac{AP}{BP} = \frac{2 - 5}{-1 - (-4)} = \frac{-3}{3} = -1.$$

If we do the reverse, taking $B(-1, 2)$ as (x_1, y_1) and $A(-4, 5)$ as (x_2, y_2) , then

$$\text{Slope} = \frac{PA}{PB} = \frac{5 - 2}{-4 - (-1)} = \frac{3}{-3} = -1.$$

Hence the slope remains the same irrespective of the order in which we choose the points.

In general, if (x_1, y_1) and (x_2, y_2) are two points on a line, the slope is given by the expression $\frac{y_2 - y_1}{x_2 - x_1}$, and it remains the same for any two points on the line. There is a simple and elegant proof of this result which you will see below.

13.3.2 Slope-Intercept Form

In the chapter on linear polynomials, we were introduced to the fact that in the linear equation $y = mx + d$, ' m ' represents the slope or steepness of the line while ' d ' represents the y-intercept, that is, the length cut off by the line on the y-axis from the origin. Thus, $y = mx + d$ is also referred to as the **slope-intercept form of a line**.

But why does 'm' represent the slope? To understand this let us consider two points $A(x_1, y_1)$ and $B(x_2, y_2)$ which lie on the line $y = mx + d$. Since they satisfy the equation we get, $y_1 = mx_1 + d$ and $y_2 = mx_2 + d$.

Now, the slope of the line passing through the points A and B is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{(mx_2 + d) - (mx_1 + d)}{x_2 - x_1} = \frac{mx_2 - mx_1}{x_2 - x_1} = \frac{m(x_2 - x_1)}{x_2 - x_1} = m$$

This is the reason why we can just read off the slope of a line if its equation is in the form $y = mx + d$. This also shows that the slope m of a line with equation $y = mx + d$ is given by $\frac{y_2 - y_1}{x_2 - x_1}$ for any two points (x_1, y_1) and (x_2, y_2) on the line, regardless of which points they are.

Note that the standard form of the equation of a line $ax + by + c = 0$ can be rewritten as

$$by = -ax - c \text{ or } y = -\frac{a}{b}x - \frac{c}{b}, (b \neq 0),$$

which is of the form $y = mx + d$. Here $m = -\frac{a}{b}$ and $d = -\frac{c}{b}$.

Hence $y = mx + d$ is another form of expressing the linear equation $ax + by + c = 0$, provided $b \neq 0$.

Consider the line $5x + y = 3$. Rewrite it as $y = -5x + 3$. Can you now find its slope? What does it mean?

In $y = -5x + 3$, $m = -5$ is the slope, which means for every increase of 1 unit in x , y decreases by 5 units. $d = 3$ is the y -intercept which means the line crosses the y -axis at $(0, 3)$.

Let us make a table of values to draw its graph.

x	0	1	-1
y	3	-2	8

Plotting the ordered pairs $A(0, 3)$, $B(1, -2)$, $C(-1, 8)$ and drawing the line through them we get the graph of $y = -5x + 3$.

Think and Reflect

Can you use the points A, B and C to verify that the slope of this line is indeed -5?

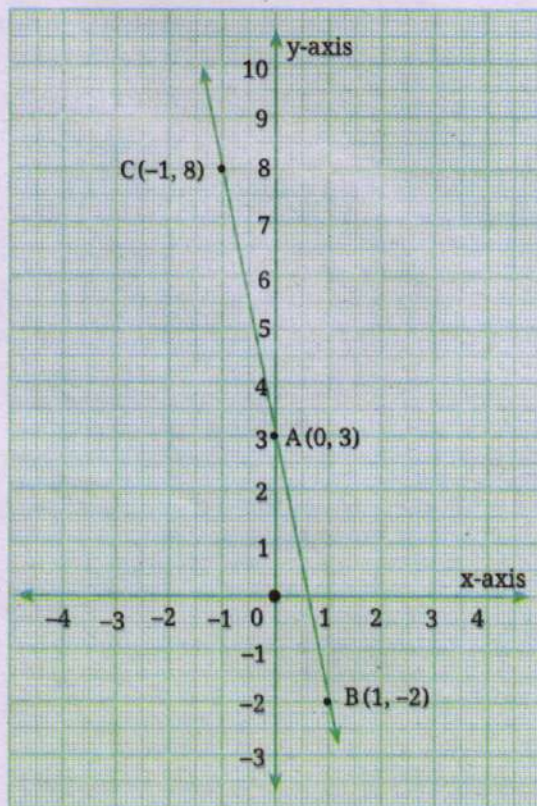


Fig. 13.11: Graph of the linear equation $y = -5x + 3$

Let us further explore the geometrical meaning of the slope of a line $y = mx + d$. To do this we will consider the following cases:

1. **Positive Slope ($m > 0$).** The line goes upward from left to right. As x increases, y increases. In $y = 2x + 1$, the slope is $m = 2$. It means for every unit you move to the right, you also go up 2 units.
2. **Negative Slope ($m < 0$).** The line goes downward from left to right. As x increases, y decreases. In $y = -3x + 4$, the slope is $m = -3$. It means for every unit you move to the right, you go down by 3 units.
3. **Zero Slope ($m = 0$).** In this case the line is horizontal and parallel to the x -axis. That is, y remains constant as x changes. For example, in $y = 5$, the slope is $m = 0$.

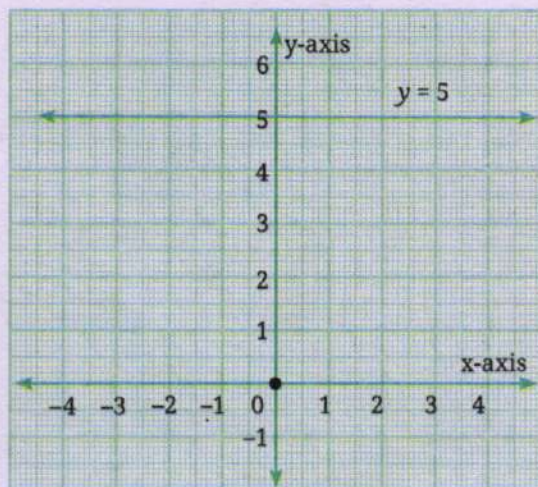


Fig. 13.12: Graph of $y = 5$

4. **Undefined Slope.** Here the line is vertical and parallel to the y-axis. x is constant as y changes. The line $x = 2$ is an example. But it is not of the form $y = mx + d$. While calculating the slope of this line we get 0 in the denominator. Since division by 0 is undefined, we take the slope also to be undefined.

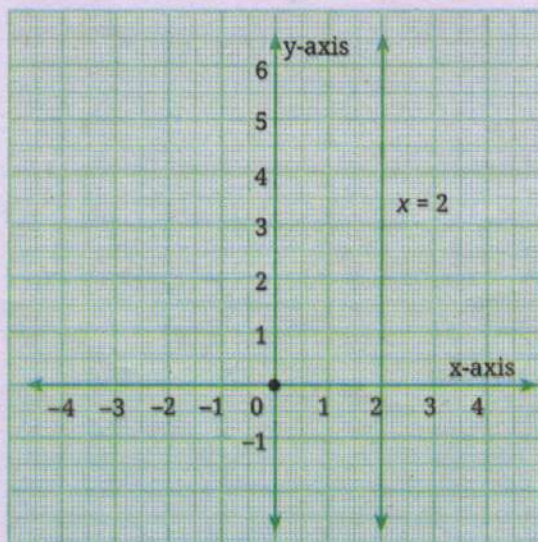
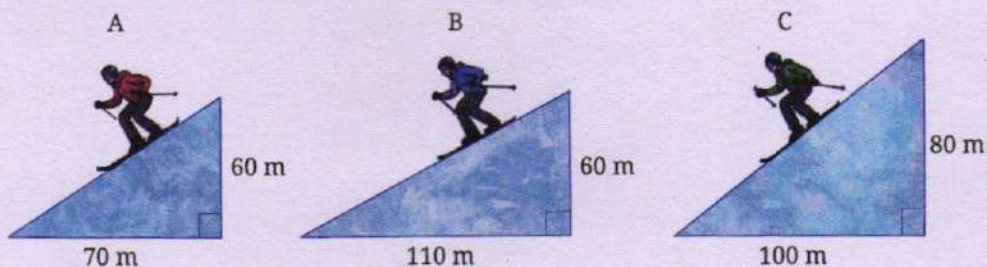


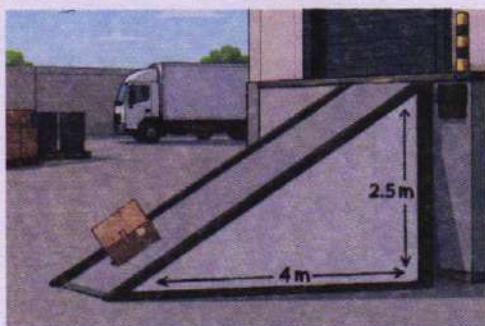
Fig. 13.13: Graph of $x = 2$

EXERCISE SET 13.3

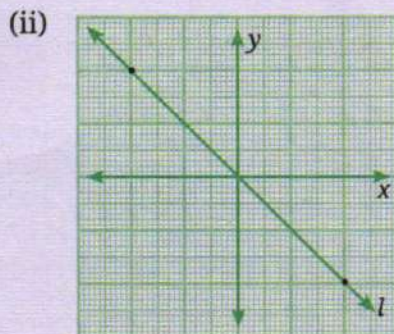
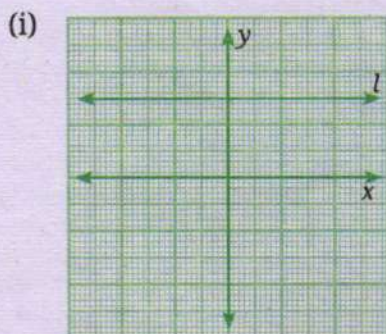
1. The following diagrams represent ski hills. Rank the hills in order of their steepness, from least to greatest.



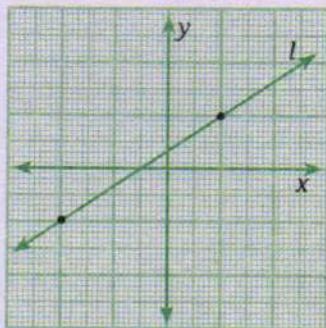
2. The ramp at a loading dock rises 2.5 metres over a run of 4 metres. Find the slope of the ramp.



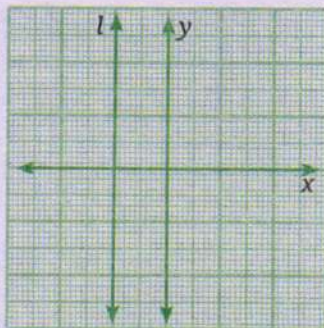
3. Find the slope of the line l in each of the following diagrams.



(iii)



(iv)



4. An accessibility ramp for wheelchairs is to be made alongside the staircase. The guidelines given for the slope of the ramp is 1 cm vertical rise to 12 cm horizontal length. What should be the horizontal length of the ramp if the total height of the stairs is 18 cm?



13.4. PAIR OF LINEAR EQUATIONS IN TWO VARIABLES

₹90 /- 1 + 2 Paratha Dahi	₹260 /- 2 + 7 Paratha Dahi
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Think and Reflect

The cost of one paratha and 2 bowls of *dahi* is ₹90. Also, the cost of 2 parathas and 7 bowls of *dahi* is ₹260. Can you figure out the cost of one paratha and one bowl of *dahi* and explain your reasoning?

Do you remember Rakesh's puzzle from Grade 7? "I have thought of two numbers", he said. "Their sum is 25 and their difference is 11". We solved the problem by making a table and listing possible combinations. We can also solve this using a pair of linear equations.

First Number	Second Number	Sum	Difference
10	15	25	-5
20	5	25	15
18	7	25	11

If the two numbers are x and y , we get the equations

$$x + y = 25 \text{ and } x - y = 11.$$

Such problems lead to a **pair of linear equations in two variables**. A solution to such a pair of equations is an ordered pair (a, b) that satisfies both equations. How do we find a solution? We shall soon see.

Consider another situation. A science exhibition charges an entry fee that is different for adults and children. Here are the amount paid by two different groups.

Group A: 2 adult tickets and 3 child tickets for ₹600

Group B: 3 adult tickets and 2 child tickets for ₹700

How much do individual adult and child tickets cost?

If the cost of one adult ticket is ₹ x and one child ticket is ₹ y , then:

$$2x + 3y = 600, 3x + 2y = 700.$$

This gives us another pair of linear equations in two variables.

A solution to these equations will give us the answer that we seek. Can you see why?

In general, a pair of linear equations in two variables x and y can be written as:

$a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, where $a_1, b_1, c_1, a_2, b_2, c_2$ are all real numbers, a_1 and b_1 are not both zero, and a_2 and b_2 are not both zero.

EXERCISE SET 13.4

Given below are some everyday situations. Construct a pair of linear equations in two variables for each situation:

- On two different days a family buys movie tickets and snack boxes. Let the cost of one movie ticket be ₹ x and the cost of one snack box be ₹ y . Frame a pair of linear equations in x and y to represent the following situations:
 - On the first day, the family buys 2 movie tickets and 3 snack boxes for ₹850.
 - On the second day, the family buys 4 movie tickets and 1 snack box for ₹1100.
- Two friends, Sahil and Meena, travelled by taxi. Let the fixed charge for one taxi trip be ₹ x and the additional charge per kilometre be ₹ y . Frame a pair of linear equations in x and y to represent the following situation:
 - Sahil travelled 6 km and paid a total fare of ₹122.
 - Meena travelled 8 km and paid a total fare of ₹160.
- In a sports meet, tickets for adults cost ₹150 each and tickets for children cost ₹100 each. A total of 200 people attended the meet, and the total amount collected from ticket sales was ₹25,000. (Let the number of adult tickets sold be x and the number of children's tickets sold be y .) Frame a pair of linear equations in x and y to represent this situation.

13.5 FINDING SOLUTIONS TO A PAIR OF LINEAR EQUATIONS

We have already seen how to solve a linear equation in a single variable. In order to find the solution of a pair of linear equations, we need to think of methods by which we can reduce the pair to a linear equation in one variable. In this section we shall study two methods.

13.5.1 Algebraic Methods for Solving a Pair of Linear Equations

Think and Reflect

Solve the puzzle and discuss your strategy with your friends. Can you determine the value represented by each shape?

				28
				30
				18
				20
?	30	23	22	

We note from the first row that the values of 2 triangles and 2 rectangles add up to 28. We may express this as $2T + 2R = 28$ or as $T + R = 14$. Similarly, the last column may be written as $2C + T + R = 22$. Here T, S and C represent the values assigned to a triangle, a rectangle and a circle respectively.

Since we know that $T + R = 14$, we can rewrite the second equation as $2C + 14 = 22$, solving which, we get $C = 4$. Once you have the value of C, you can find the values of the other shapes. Try it for yourself.

We will now learn two methods for solving a pair of linear equations. One is called the **method of substitution** while the other is referred to as the **method of elimination**.

Substitution Method: We will demonstrate the method of substitution by solving a pair of linear equations in two variables.

Example 7: Solve the following pair of equations by substitution method:

$$7x - 15y = 2 \quad (1)$$

$$x + 2y = 3 \quad (2)$$

Solution:

Step 1: We select either of the equations and express one variable in terms of the other.

Let us consider Equation (2), that is, $x + 2y = 3$ and write it as

$$x = 3 - 2y \quad (3)$$

Step 2: Clearly, all (x, y) that satisfy both the given equations must also satisfy $x = 3 - 2y$. Since this relation must hold true in Equation (1), substituting $x = 3 - 2y$ in it we get

$$7(3 - 2y) - 15y = 2$$

$$\begin{array}{rcl} \text{or} & 21 - 14y - 15y & = 2 \\ \text{or} & -29y & = -19 \end{array}$$

$$\text{Therefore, } y = \frac{19}{29}.$$

Step 3: We can find the value of x by substituting the value of y in any of the equations. Let us use Equation (3).

$$x = 3 - 2\left(\frac{19}{29}\right) = \frac{49}{29}.$$

Therefore, the solution of the Equations (1) and (2) is $x = \frac{49}{29}, y = \frac{19}{29}$.

To verify if the solution is correct, we substitute $x = \frac{49}{29}, y = \frac{19}{29}$, in Equations (1) and (2) and check if they are satisfied. Try it for yourself.

Think and Reflect

What if we had expressed y in terms of x , i.e., take Equation (2) and write $y = \frac{1}{2}(3 - x)$. Would we still get the same solution?

Example 8: If two angles of a triangle are x° and y° with $y = 4x$, and the third angle is 50° , what are the angles?

The three angles of the triangle are x°, y° , and 50° .

Using the angle sum property of triangles,

$$x^\circ + y^\circ + 50^\circ = 180^\circ \quad (1)$$

$$\text{Also,} \quad y = 4x. \quad (2)$$

Substituting the value of y in Equation (1),

$$\text{we get} \quad x + 4x + 50 = 180$$

$$\text{or} \quad 5x = 130$$

$$\text{or} \quad x = 26^\circ.$$

$$\text{Therefore, } y = 4x = 104^\circ.$$

Elimination Method

Now let us consider the method of elimination for solving a pair of linear equations. This requires us to eliminate one of the variables x or y . Sometimes it is more convenient than the substitution method. Let us see how this method works.

Example 9: The ratio of incomes of two persons is 9 : 7 and the ratio of their expenditures is 4 : 3. If each of them manages to save ₹2000 per month, find their monthly incomes.

Since the ratio of the incomes of the two persons is 9 : 7, it means that for every 9 equal parts of income of the first person, the second person has 7 equal parts. If the equal part is x , the incomes of the two persons will be $9x$ and $7x$.

In a similar manner, we may consider their expenditures to be $4y$ and $3y$, respectively.

Then the equations formed by subtracting the expenditure from the income leading to a saving of ₹ 2000 are as follows:

$$9x - 4y = 2000 \quad (1)$$

$$7x - 3y = 2000 \quad (2)$$

To solve these by the method of elimination, we proceed as follows:

Step 1: We first decide which variable to eliminate. Suppose we wish to eliminate y from both equations. We multiply each side of Equation (1) by 3 [the coefficient of y in Equation (2)], and multiply Equation (2) by 4 [the coefficient of y in (1)]. By doing this the coefficients of y in both equations become equal. We get the equations:

$$27x - 12y = 6000 \quad (3)$$

$$28x - 12y = 8000 \quad (4)$$

Step 2: Subtract the LHS and RHS of Equation 3 from the LHS and RHS of Equation 4 respectively. Since we are subtracting equal quantities from equal quantities, we get

$$(28x - 27x) - (12y - 12y) = 2000.$$

This step eliminates the variable y , leaving us with $(28x - 27x) = 2000$ or $x = 2000$.

Step 3: Substituting this value of x in (1), we get

$$9(2000) - 4y = 2000 \text{ i.e., } y = 4000.$$

Hence, the solution of the equations is $x = 2000$, $y = 4000$.

Therefore, the monthly incomes of the two persons are ₹18,000 and ₹14,000, respectively.

Let us verify our solution. The ratio of the incomes is $18000 : 14000 = 9 : 7$. Also, the ratio of their expenditures = $(18000 - 2000) : (14000 - 2000) = 16000 : 12000 = 4 : 3$. Hence, the conditions of the problem are satisfied.

Think and Reflect

The method used in solving this problem is called the Elimination Method because we eliminate one of the variables to obtain a linear equation in the other variable. In the example above, we eliminated y . Try the same problem by eliminating x instead of y . Also try to solve the two equations using the Substitution Method. Assess the pros and cons of each method.

Example 10: The sum of a two-digit number and the number obtained by reversing its digits is 66. If the digits of the number differ by 2, find the number. How many such numbers are there?

Let the ten's and the unit's digits in the first number be x and y , respectively. Also let us assume that x is the larger digit and y is the smaller one. So, the first number may be written as $10x + y$ in the expanded form (for example, $73 = 10 \times 7 + 3$).

When the digits are reversed, x becomes the unit's digit and y becomes the ten's digit. This number, in the expanded notation, is $10y + x$ (for example, when 73 is reversed, we get $37 = 10 \times 3 + 7$).

According to the given condition, we have $(10x + y) + (10y + x) = 66$. Thus

$$11(x + y) = 66, \text{ i.e., } x + y = 6 \quad (1)$$

We are also given that the digits differ by 2, i.e.,

$$x - y = 2 \quad (2)$$

Solving (1) and (2) by elimination, we get $x = 4$ and $y = 2$, so, the two-digit number is 42.

The number 42 and its reverse 24 when added gives 66, and this is the unique solution.

Think and Reflect

Does a pair of linear equations always have a unique solution? Can you analyse this using the Elimination Method? While subtracting the equations to eliminate a particular variable, if the other variable remains, then clearly there is a unique solution. But what if the second variable gets eliminated too while eliminating the first variable? Can you think of a situation where this happens?

Consider the linear equations

$$x - y = 10 \quad (1)$$

$$10x - 10y = 100 \quad (2)$$

To eliminate x , we multiply the first equation by 10. But by doing this, both equations become $10x - 10y = 100$. Since both equations are the same, every solution of $10x - 10y = 100$ is a solution of both equations. We know that $10x - 10y = 100$ has infinitely many solutions. Hence, the given pair of equations has infinitely many solutions.

Think and Reflect

How many solutions are there to the equations

$$10x - 5y = 20 \text{ and } 4x - 2y = 8?$$

Give 3 more examples of pairs of equations that lead to $0 = 0$ after subtraction, and therefore have infinitely many solutions. Can you find a simple rule to check when this happens?

Consider a general pair of equations

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

with a_2, b_2 and c_2 each not equal to 0.

From the discussion above, it can be seen that whenever one of the equations is a constant multiple of another, the equations are the same and so have an infinite number of solutions. Suppose this constant multiple is k , where k is not necessarily an integer but may be any number. Then

$$a_1 = ka_2, b_1 = kb_2, c_1 = kc_2, \text{ where } k \neq 0.$$

Thus, when $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = k$, the pair of equations has infinitely many solutions.

Think and Reflect

What if one of a_2, b_2, c_2 is zero?

Think and Reflect

Does a pair of linear equations always have either a unique solution or infinitely many solutions?

Consider the linear equations

$$x - y = 10 \quad (1)$$

$$10x - 10y = 101 \quad (2)$$

Multiplying both sides of (1) by 10 gives

$$10x - 10y = 100 \quad (3)$$

The left hand side in (2) and (3) are equal expressions, but the right hand side are not. Hence we cannot find values of x and y that satisfy both (2) and (3). Thus, this pair of equations has no solution!

Think and Reflect

Give 3 more examples of pairs of equations that have no solution. Can you find a simple rule to check when this will happen?

When a pair of equations

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

(with a_2 , b_2 and c_2 all not equal to 0) is such that after suitable multiplications, we can make

(i) the coefficients of x of both equations equal, and

(ii) the coefficients of y of both equations equal

while the constants c_1 and c_2 remain unequal, the pair has no solution.

These conditions mean $a_1 = ka_2$, $b_1 = kb_2$, $c_1 \neq kc_2$, where $k \neq 0$.

Hence, when $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, the pair has no solution.

When $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, both the variables get eliminated during the Elimination Method. Based on what happens to the constants, the pair can either have an infinite number of solutions or no solution.

We have already seen that when one of the variables remains during the Elimination Method, there exists a unique solution. This can happen only when $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.

Summarising, we have the following cases:

1. when $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, the pair has a unique solution,

2. when $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, the pair has an infinite number of solutions, and
3. when $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, the pair has no solution.

Think and Reflect

Are the converses of the above statements true?

Thus, (1), (2) and (3) can be used as simple rules to check for the nature of solutions.

Think and Reflect

Can there be methods other than elimination and substitution to reduce a pair of linear equations in two variables to a linear equation in one variable? If so, describe them.

A Pinch of History

Solving systems of linear equations is one of the oldest problems in mathematics, going back thousands of years. The earliest traces appear in Babylonian cuneiform tablets (c. 2000 BCE). They were studied systematically in ancient China; the Nine Chapters on the Mathematical Art (compiled roughly two thousand years ago) devotes an entire chapter, fangcheng, to systems of linear equations, solved by arranging coefficients in columns of counting rods and then combining columns to cancel out variables—a procedure essentially identical to modern elimination, fully eighteen centuries before Gauss's name became attached to it in Europe. Many Indian mathematicians, including Āryabhaṭa (499 CE) and Brahmagupta (628 CE), also studied general systems of equations in several unknowns using their own systematic reduction techniques including elimination.

13.6 GRAPHICAL METHOD FOR SOLVING A PAIR OF LINEAR EQUATIONS IN TWO VARIABLES

In this section, we learn how to solve a pair of linear equations graphically, that is by plotting the lines represented by the two equations.

Consider the following pairs of linear equations in two variables.

Pair 1: $2x + y = 6$ and $x - y = 2$

Pair 2: $x + y = 4$ and $2x + 2y = 8$

Pair 3: $3x - 2y = 6$ and $6x - 4y = 12$

Pair 4: $x - 2y = 4$ and $2x - 4y = 6$

For each pair, prepare a table of values (with at least two ordered pairs). Plot the points on the Cartesian plane. Draw the straight lines.

Think and Reflect

How do we find solutions from the graph? What can we say about the number of solutions when the two lines (i) intersect at a point, (ii) are parallel, (iii) are coincident?

Solutions to the pair of equations are points that lie on both lines.

Thus

- (i) when the lines intersect in a single point, there is a unique solution;
- (ii) when the lines are parallel, there are no solutions; and
- (iii) when the lines are coincident, there are infinitely many solutions.

Example 11: Solve the pair of linear equations $x + 3y = 6$ and $2x - 3y = 12$ graphically. Comment on the nature of the solution.

Consider the equation $x + 3y = 6$. Expressing y in terms of x , we get

$$y = \frac{6-x}{3}.$$

Considering x to be first 0 and then 6 we obtain the corresponding values of y as 2 and 0 respectively. Hence (0, 2) and (6, 0) are points on the line.

x	0	6
y	2	0

Now consider the equation $2x - 3y = 12$. Expressing y in terms of x , we get $y = \frac{2x-12}{3}$. Considering x to be 0 and then 3, we obtain the values of y as -4, and -2 respectively. Hence (0, -4) and (3, -2) are points on the line.

x	0	3
y	-4	-2

Now plot the points A(0, 2), B(6, 0), P(0, -4) and Q(3, -2) on a graph paper, and join the points to form the lines AB and PQ as shown in Fig. 13.14. We observe that the point B(6, 0) is common to both the lines. In fact the lines AB and PQ intersect at B. So, the solution of the pair of linear equations is $x = 6$ and $y = 0$. Also, we can say that the given pair of equations has a **unique solution**.

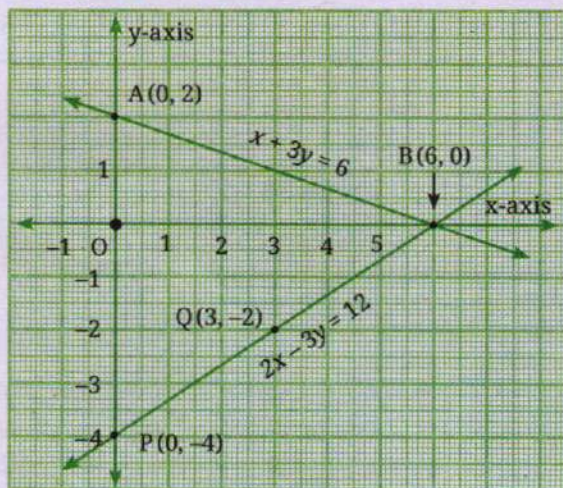


Fig. 13.14: Graph of the pair of linear equations $x + 3y = 6$ and $2x - 3y = 12$. They have the unique solution $(6, 0)$ or $x = 6, y = 0$

If we consider the ratios of the corresponding coefficients and constants in the equations $x + 3y = 6$ and $2x - 3y = 12$, we get $\frac{a_1}{a_2} = \frac{1}{2}$, $\frac{b_1}{b_2} = \frac{3}{-3} = -1$. Here $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$. We have also seen graphically that the lines intersect.

Example 12: Determine if the lines $x + 2y - 4 = 0$ and $2x + 4y - 12 = 0$ intersect, coincide or are parallel to each other. Then verify your answer by graphing the equations.

On comparing the coefficients of these two equations we get $\frac{a_1}{a_2} = \frac{1}{2}$, $\frac{b_1}{b_2} = \frac{2}{4} = \frac{1}{2}$, $\frac{c_1}{c_2} = \frac{-4}{-12} = \frac{1}{3}$. We observe that $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ but is not equal to $\frac{c_1}{c_2}$. Thus, the equations have no solution and hence the lines are parallel.

As before, we identify the points (0, 2) and (4, 0) on the line $x + 2y - 4 = 0$

x	0	4
y	2	0

Similarly we find two points on the line $2x + 4y - 12 = 0$, say, $(0, 3)$ and $(6, 0)$.

x	0	6
y	3	0

To represent the equations graphically, we plot the points $R(0, 2)$ and $S(4, 0)$, to get the line RS and the points $P(0, 3)$ and $Q(6, 0)$ to get the line PQ .

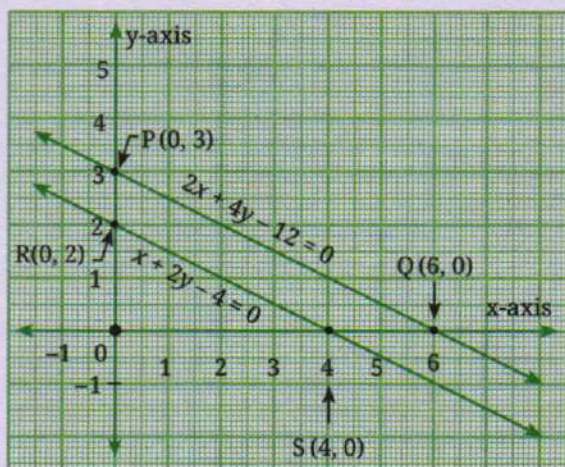


Fig. 13.15: Graph of the pair of linear equations $2x + 4y - 12 = 0$ and $x + 2y - 4 = 0$

We observe in Fig. 13.15 that the lines do not intersect anywhere and are parallel.

Think and Reflect

Can you prove that two lines of equal slope are parallel?

(Hint: Consider the equations of the two lines to be $y = mx + d_1$ and $y = mx + d_2$)

Example 13: Romila went to a stationery shop and purchased 2 erasers and 3 A4 sheets for ₹9. Her friend Sonali saw the new variety of erasers and A4 sheets Romila bought, and she also bought 4 erasers and 6 A4 sheets of the same kind for ₹18. Represent this situation graphically.

Let us denote the cost of a erasers by ₹ x and one A4 sheets by ₹ y . Then the algebraic representation is given by the equations $2x + 3y = 9$ and $4x + 6y = 18$. Writing these equations in standard form we get $2x + 3y - 9 = 0$ and $4x + 6y - 18 = 0$.

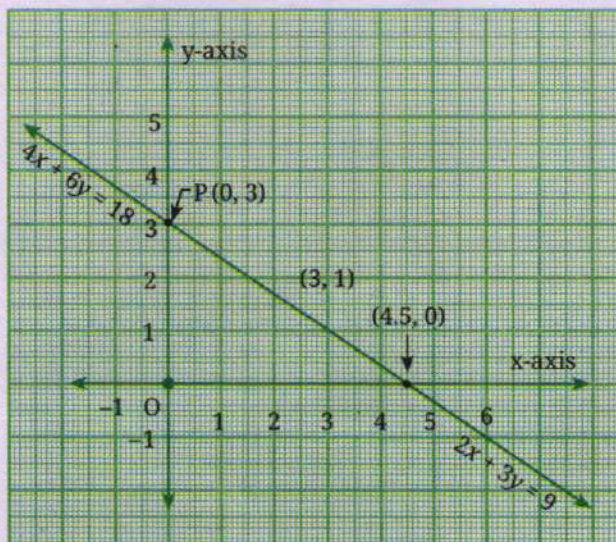


Fig. 13.16: Graphs of the pair of linear equations $2x + 3y = 9$ and $4x + 6y = 18$. The two lines coincide.

We find that both equations are represented by the same straight line. On comparing the coefficients of the two equations we find that $\frac{a_1}{a_2} = \frac{2}{4} = \frac{1}{2}$, $\frac{b_1}{b_2} = \frac{3}{6} = \frac{1}{2}$, $\frac{c_1}{c_2} = \frac{-9}{-18} = \frac{1}{2}$ which represents the fact that the pair of lines has infinitely many solutions.

EXERCISE SET 13.5

1. Form a pair of linear equations for each of the following problems and find their solutions.
 - (i) The sum of two integers is +5 and their difference is -21. Find the two numbers.
 - (ii) The difference between two numbers is 26 and one number is three times the other. Find the numbers.
 - (iii) The coach of a cricket team buys 7 bats and 6 balls for ₹8880. Later, she buys 3 bats and 5 balls for ₹4000. Find the cost of each bat and each ball.
 - (iv) The taxi charges in a city consist of a fixed charge together with the charge for the distance covered for every km. For a distance of 10 km, the total amount paid is ₹155 and for

a journey of 15 km, the total amount paid is ₹220. What is the fixed charge and the charge per km? How much does a person have to pay for travelling a distance of 25 km?

- (v) A fraction becomes equal to $\frac{9}{11}$ if 2 is added to both the numerator and the denominator. If 3 is added to both the numerator and the denominator, it becomes equal to $\frac{5}{6}$. Find the fraction.
 - (vi) If we add 1 to the numerator and subtract 1 from the denominator, a fraction reduces to 1. It becomes equal to $\frac{1}{2}$ if we add 1 only to the denominator. What is the fraction?
 - (vii) Five years ago, Nuri was thrice as old as Sonu. Ten years later, Nuri will be twice as old as Sonu. How old are Nuri and Sonu?
 - (viii) The sum of the digits of a two-digit number is 9. Also, nine times this number is twice the number obtained by reversing the order of the digits. Find the number.
 - (ix) Meena went to a bank to withdraw ₹2000. She asked the cashier to give her ₹50 and ₹100 notes only. Meena got 25 notes in all. Find how many notes of ₹50 and ₹100 did she receive?
 - (x) A lending library has a fixed charge for the first three days and an additional charge for each day thereafter. Saritha paid ₹27 for a book kept for seven days, while Susy paid ₹21 for the book she kept for five days. Find the fixed charge and the charge for each extra day.
2. Form a pair of linear equations and find their common solutions graphically.
- 10 students of Grade 9 took part in a Mathematics quiz. If the number of girls is 4 more than the number of boys, find the number of boys and girls who took part in the quiz.
3. Use the ratios $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$, $\frac{c_1}{c_2}$, to determine whether the lines representing the following pairs of linear equations intersect at a point, are parallel or are coincident.
- (i) $5x - 4y + 8 = 0$; $7x + 6y - 9 = 0$
 - (ii) $9x + 3y + 12 = 0$; $18x + 6y + 24 = 0$
 - (iii) $6x - 3y + 10 = 0$; $2x - y + 9 = 0$

4. Which of the following pairs of linear equations have solutions? If they have solutions, find them graphically.

- (i) $x + y = 5$, $2x + 2y = 10$
 (ii) $x - y = 8$, $3x - 3y = 16$
 (iii) $2x + y - 6 = 0$, $4x - 2y - 4 = 0$
 (iv) $2x - 2y - 2 = 0$, $4x - 4y - 5 = 0$

5. Half the perimeter of a rectangular garden, whose length is 4 m more than its width, is 36 m. Find the dimensions of the garden.
6. Given the linear equation $2x + 3y - 8 = 0$, write another linear equation in two variables so that the graphs of the pair formed represent (i) intersecting lines (ii) parallel lines (iii) coincident lines.
7. Here is a problem that was posed by Mahāvīrāchārya in *Gaṇita sāra saṅgraha* (c. 850 CE).

The price of 9 citrons and 7 fragrant wood-apples taken together is 107; and the price of 7 citrons and 9 fragrant wood-apples taken together is 101.

O mathematician, tell me quickly the price of each citron and of each fragrant wood-apple.

(Hint: The given problem can be modelled as

$$9x + 7y = 107$$

$$7x + 9y = 101$$

Can you figure out a way of solving these equations without directly using elimination or the substitution method? What is special in this pair of equations? The x and y coefficients are interchanged in the 2 equations.

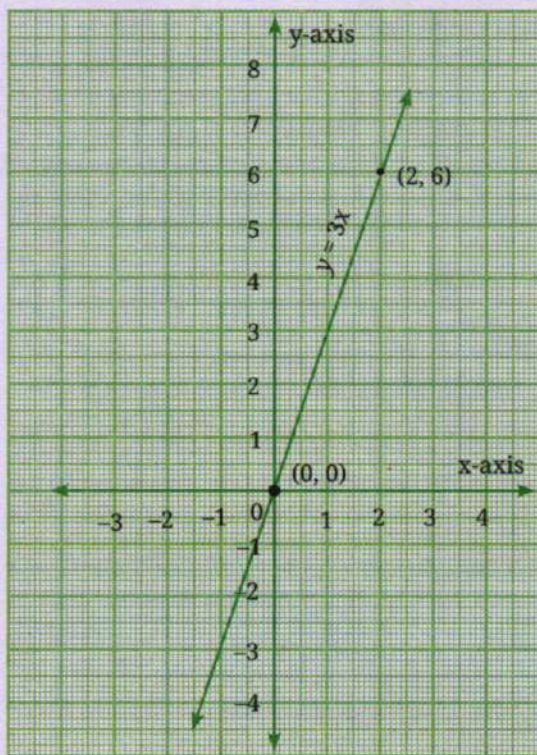
What will happen if we add the pair of equations? What will happen if we subtract the pair of equations? Can the resulting equations be solved to find the values of x and y ?

8. 5 pencils and 7 pens together cost ₹50, whereas 7 pencils and 5 pens together cost ₹46. Find the cost of each pencil and pen.
9. Find the height of the stool.



END-OF-CHAPTER EXERCISES

1. The graph of the line $y = 3x$, passing through $(0, 0)$ and $(2, 6)$ is given below.
- (i) Identify the slope of the line from the graph and explain how you calculated it using the two points on the line.
 - (ii) What is the y-intercept of the line? Explain its significance.
 - (iii) A water tank is being filled so that the water level y (in cm) after x minutes follows this graph.



- (a) How high is the water after 5 minutes?
 - (b) How many minutes will it take for the water to reach a height of 21 cm?
- (iv) Without drawing a new graph, determine whether the point $(4, 10)$ lies on this line. Justify your answer.
- (v) Plot the point where this line intersects the line $x = 3$. Explain how you found the coordinates.

2. In countries like the USA, temperature is measured in Fahrenheit, whereas in countries like India, it is measured in Celsius. Here is a linear equation that converts Celsius to Fahrenheit:

$$F = \frac{9}{5}C + 32$$

- (i) Draw the graph of the linear equation above using Celsius on the x-axis and Fahrenheit on the y-axis.
- (ii) If the temperature is 30°C , what is the temperature in Fahrenheit?
- (iii) If the temperature is 95°F , what is the temperature in Celsius?
- (iv) If the temperature is 0°C , what is the temperature in Fahrenheit and if the temperature is 0°F , what is the temperature in Celsius?
- (v) Is there a temperature which is numerically the same in both Fahrenheit and Celsius? If yes, find it.

3. Solve the following system of equations graphically:

$$2x + y = 6, 2x - y - 2 = 0.$$

4. Find the point of intersection of the lines shown on the cover page.
5. Give a formula to find the x-intercept of the line $y = mx + c$.
6. A person is choosing between two mobile plans.
Plan A: ₹50 monthly fee + ₹0.20 per minute of call time.
Plan B: ₹30 monthly fee + ₹0.30 per minute of call time.
For how many minutes of calling per month is Plan A cheaper than Plan B? For how many minutes is Plan B cheaper? Also find the number of minutes at which both plans cost the same.
7. How many lines exist that
 - (i) have a given slope?
 - (ii) have a given slope and pass through a given point?
8. For what values of p does the pair of equations given below have a unique solution?

$$4x + py + 8 = 0; \quad 2x + 2y + 2 = 0.$$

9. Find the values of a and b for which the following system of equations has infinitely many solutions:

$$(a + b)x - 2by = 5a + 2b + 1; 3x - y = 14.$$

10. Find the value of ' k ' for which the following system of equations represents a pair of coincident lines:

$$x + 2y = 3; (k - 1)x + (k + 1)y = k + 3.$$

11. (i) Robot 1 starts from the origin and traces a path by repeatedly moving 3 units to the right and then 4 units upward. Robot 2 starts from the point (10, 0) and repeatedly moves 1 unit to the right and then 2 units upward. Will the paths traced by these two robots intersect and if so, where?
- (ii) Robot 1 starts from (3, 0) and traces a path by repeatedly moving 5 units to the right and then 5 units upward. Robot 2 starts from the point (7, 0) and repeatedly moves 5 units to the right and then 3 units downwards. Will the paths traced by these two robots intersect and if so, where?
- *12. At a certain time, Jacob notices that his digital watch reads ' a ' minutes after two o'clock. Fifteen minutes later, it reads ' b ' minutes after three o'clock. He noticed that a is six times greater than b . What time was it when he looked at his watch for the second time?
- *13. The sum of the digits of a two-digit number is 15. The number obtained by interchanging the digits exceeds the given number by 9. Find the number.
- *14. In a cyclic quadrilateral ABCD, $\angle A = (x + 7)^\circ$, $\angle B = (y + 8)^\circ$, $\angle C = (3y + 23)^\circ$ and $\angle D = (4x + 12)^\circ$. Find all four angles of the cyclic quadrilateral.
- *15. A train moving with uniform speed for a certain distance takes 6 hours less if its speed is increased by 6 km/hour. It would have taken 6 hours more had its speed been decreased by 4 km/hour. Find the distance travelled and the speed of the train.
- *16. The age of a father is equal to the sum of the ages of his four children. After 20 years, the sum of the ages of the children will be twice the age of the father. Find the age of the father.

CHAPTER SUMMARY

- An equation of the form $ax + by + c = 0$, where a , b and c are real numbers, such that a and b are not both zero, is called a **linear equation in two variables**.
- A linear equation in two variables has infinitely many solutions.
- An equation $y = mx + d$ is the **slope-intercept form** of a linear equation in two variables. Here ' m ' represents the slope and ' d ' the y-intercept. The line cuts the y-axis in the point $(0, d)$.
- Every point on the graph of a linear equation in two variables is a solution of the linear equation. Moreover, every solution of the linear equation is a point on the graph of the linear equation.
- A pair of linear equations in two variables can be represented, and solved, both algebraically and graphically.
- A pair of linear equations can be solved algebraically using the Method of Substitution or the Method of Elimination.
- If a pair of linear equations is given by $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then the following cases may arise:
 - $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$: In this case, the pair of linear equations **has a unique solution**.
 - $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$: In this case, the pair of linear equations **does not have a solution**.
 - $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$: In this case, the pair of linear equations **has infinitely many solutions**.
- The graph of a pair of linear equations in two variables is represented by two lines.
 - If the lines intersect at a point, then that point is the **unique solution** of the two equations.
 - If the lines coincide, then there are **infinitely many solutions**, each point on the line being a solution.
 - If the lines are parallel, then the pair of equations has **no solution**.



0905CH14

In the previous grade we explored nets of some solids including cubes, cylinders, and cones. We saw that solids have a surface and occupy some space or volume. In this chapter, we will learn how to determine surface areas and volumes of cuboids, cylinders, cones, pyramids, and spheres, and how we can use these to solve some interesting problems.

14.1 CUBOIDS AND CUBES

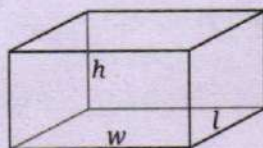
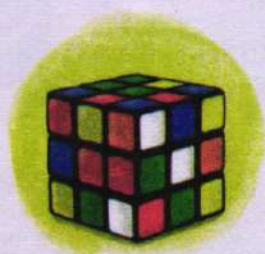


Fig. 14.1A: Cuboid bricks Fig. 14.1B: Rubik's Cube Fig. 14.1C: Cuboid

A *cuboid* has the shape shown in these pictures. A useful way of thinking about a cuboid is that it is a 'three-dimensional (3D) version' of a rectangle.

It has six faces, all rectangles. It is characterised by three quantities: length, width, and height — these are denoted by l , w , h respectively. You can cut out the net of a cuboid given at the end of the textbook, make appropriate folds and join the edges to form the cuboid.

What is the total surface area of a cuboid? You may refer to the net of the cuboid given in Fig. 14.2.

Considering the faces individually, we see that the total surface area (TSA) of the cuboid is

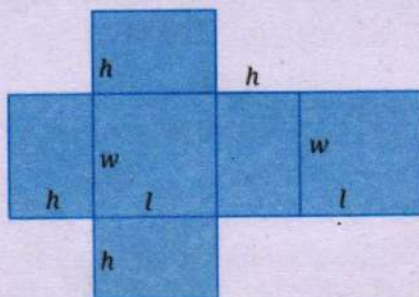


Fig. 14.2

$$\text{TSA} = 2(wl + hl + hw).$$

Volume measures the amount of three-dimensional space an object or a container occupies. It is measured in cubic units, e.g., cubic meters (m^3) or cubic centimeters (cm^3). We set the volume of a *unit cube* (i.e., a cuboid measuring one unit in each direction) to be one cubic unit (e.g., $1 m^3$ or $1 cm^3$, as the case may be). The volume of a general cuboid is the number of unit cubes that fit into it. Accordingly, the formula for the volume V of a cuboid is length $\times$ width $\times$ height or

$$V = lwh.$$

Compare the formula for the volume of a cuboid with the formula for the area of a rectangle (length $\times$ width).

Here is a way of understanding why the formula $V = lwh$ is true: think of a cuboid as a large number of thin rectangular pieces (length l , width w) stacked on top of each other, like a pack of playing cards. (Fig. 14.3).



Fig. 14.3

Think and Reflect

Try to work out for yourself why this model explains the formula for the volume of a cuboid, i.e.,

$$\text{volume} = \text{area of base} \times \text{height}.$$

Write the formula for the surface area and the volume of a cube.

A particular and important case of a cuboid is a *cube*, whose length, width, and height are the same. This length is the 'side' of the cube.

Let the side be a . Then:

- Total surface area of the cube is $6a^2$.
- Volume of the cube is a^3 .

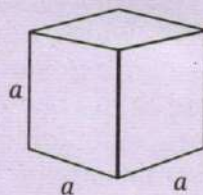


Fig. 14.4

Observe how these two formulas are special cases of the formulas $2(wl + hl + hw)$ and whl , when $w = h = l$.

Do you remember cube numbers? Cube numbers are the numbers in the sequence 1, 8, 27, 64, ... These numbers represent the number of unit cubes that can fit inside cubes of side lengths 1, 2, 3, 4, ... respectively—in other words, they represent the volumes of these cubes. We know that $1^3 = 1$, $2^3 = 8$, $3^3 = 27$, $4^3 = 64$, ...; and $\sqrt[3]{1} = 1$, $\sqrt[3]{8} = 2$, $\sqrt[3]{27} = 3$, $\sqrt[3]{64} = 4$, and so on.

Example 1: Two solid objects are made from the same material: Cube A has side 6 cm and Cuboid B has dimensions 9 cm $\times$ 6 cm $\times$ 4 cm. Compare their volumes and surface areas and determine which object has a greater surface area. How is this useful in a real-life situation?

For Cube A, side (a) = 6 cm.

Hence its volume is $a^3 = (6 \text{ cm})^3 = 216 \text{ cm}^3$ and surface area (SA) is $6a^2 = 6 \times (6 \text{ cm})^2 = 216 \text{ cm}^2$.

For Cuboid B, length (l) = 9 cm; width (w) = 6 cm; height (h) = 4 cm.

Hence its volume is $l \times w \times h = 9 \times 6 \times 4 = 216 \text{ cm}^3$

and surface area (SA) is $2(lw + wh + lh) = 2[(9 \times 6) + (6 \times 4) + (4 \times 9)] = 228 \text{ cm}^2$.

Thus, the two objects have the same volume but different surface areas. The cuboid has greater surface area than the cube.

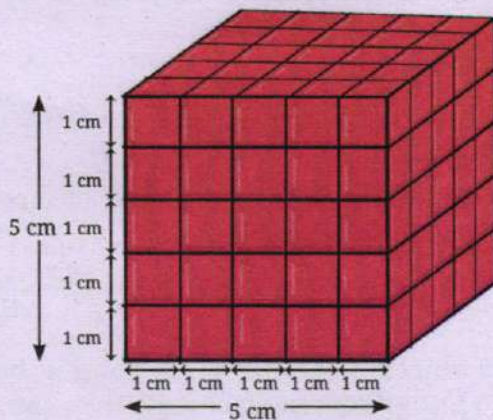
Conclusion: For the same storage capacity: a cube requires less covering material; a cuboid exposes more area to its surroundings.

This concept is important in packaging, heat transfer and engineering.

EXERCISE SET 14.1

1. The volume of a cube is 64 cm^3 . What is its total surface area?
2. How many small cubes with side 20 cm can be packed tight in a cubical box with side 2 m?
3. The dimensions of a godown are 40 m $\times$ 25 m $\times$ 10 m. If it is filled with cuboidal boxes, each of dimensions 2 m $\times$ 1.25 m $\times$ 1 m, then find the number of boxes.
4. Two cubes each of volume 125 cm^3 are joined end to end. Find the surface area of the resulting cuboid.

5. A cube of side 4 cm is cut into cubes of side 1 cm. What is the ratio of the surface areas of the original cube and all the cut-out cubes? (Note that there is no change in volume but a big change in the surface area. This property has major consequences in the biological world.)
6. The surface areas of the three faces of a cuboid that meet at one of the corners of the cuboid are 6 cm^2 , 15 cm^2 , and 10 cm^2 respectively. What is the volume of the cuboid?
- *7. A cube of side 5 cm is painted on all its faces. If it is sliced into 1 cm^3 cubes, how many of these 1 cm^3 cubes have



- (i) exactly three faces painted?
 - (ii) exactly two faces painted?
 - (iii) exactly one face painted?
 - (iv) no face painted?
- *8. Find a cuboid with edges whose lengths are integers (in cm), given that it has a total surface area of exactly 100 cm^2 .
 - (i) Is there more than one such cuboid?
 - (ii) Can you find them all?
 - (iii) Show that you have found them all.

Think and Reflect

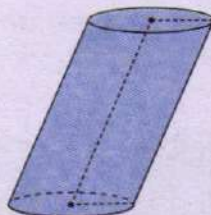
Suppose a swimming pool is being built in your school. Two choices are available for the dimensions of the pool: (A) 8 ft (depth) $\times$ 50 ft $\times$ 20 ft and (B) 6 ft (depth) $\times$ 100 ft $\times$ 15 ft. Which option would be suitable/appropriate for your school? Why? What parameters/aspects did you consider to make the choice? [1 cubic foot $\approx$ 28.3 litres]

14.2 RIGHT CIRCULAR CYLINDER

The following figures show different types of cylinders. What is different in each of these cylinders?



Right circular cylinder



Oblique circular cylinder

The first one is a **right circular cylinder**. The term *circular* indicates that the cross-section of the cylinder is a circle, while the term *right* signifies that the axis of the cylinder is perpendicular to its circular base. Many pipes have this shape. It is worth noting that not all cylinders are of this type. Some cylinders may have axes that are inclined rather than perpendicular to the base. The second one is an **oblique circular cylinder**. We restrict ourselves to the study of the right circular cylinder only. You can cut out the net of a cylinder given at the end of the textbook, make appropriate folds and join the circles to form the cylinder.

We used the word **cross-section** above. Given a three-dimensional object, suppose we take an extremely thin slice of it. The shape we get is called a 'cross-section' of the object. In the case of a cylinder, if the slice is parallel to its base, we get a circular cross-section. But if the slice is parallel to the axis of the cylinder, we get a rectangular cross-section. Generally, we are interested in the cross-section that is parallel to the base.

Examples of cross-sections are shown in the following figure.



Cylinder



Papaya



Apple

When doctors take an MRI scan of some part of your body, it is like taking a cross-section because it allows them to look at the inside of your body in slices.

What is the area of the curved surface of a cylinder?

Let the radius of the circular base of the cylinder be r ; let its height be h . Suppose the cylinder is cut as shown in the figure below. We get a rectangle.

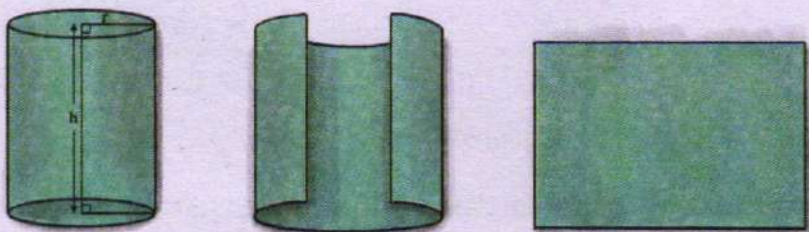


Fig. 14.5A

Therefore, the curved surface area of the cylinder is the area of the rectangle.

- The curved surface area (CSA) of the cylinder is

$$\text{CSA} = \text{circumference} \times \text{height} = 2\pi rh.$$

If the cylinder is closed at both ends (top and bottom) like a biscuit tin (see Fig. 14.5B), then the flat circular surfaces at the ends also contribute to the area. Since there are two ends,

$$\begin{aligned} \text{total surface area (TSA)} &= 2\pi rh + 2\pi r^2 \\ &= 2\pi r(h + r). \end{aligned}$$

- If the cylinder is closed at one end and open at the other end, like a tumbler, then

$$\text{surface area} = 2\pi rh + \pi r^2 = \pi r(2h + r).$$

- The volume V of the cylinder is

$$V = \text{cross-sectional area} \times \text{height} = \pi r^2 h.$$



Fig. 14.5B

14.2.1 Another Way of Understanding Why the Above Formulas Are True

To understand why the formulas for the surface area (curved part) and the volume of a cylinder hold, it helps to think of the cylinder as a large number of thin circular khakhra stacked on one another.



Example 2: Savitri had to make a model of a cylindrical kaleidoscope for her science project. She wanted to use chart paper to make the curved surface of the kaleidoscope (see Fig. 14.6). What would be the area of chart paper required by her, if she wanted to make a kaleidoscope of length 25 cm with a 3.5 cm radius? You may take $\pi = \frac{22}{7}$.

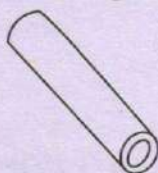


Fig. 14.6

Radius of the base of the cylindrical kaleidoscope (r) = 3.5 cm.

Height (length) of kaleidoscope (h) = 25 cm.

$$\begin{aligned}
 \text{Area of chart paper required} &= \text{curved surface area of the kaleidoscope} \\
 &= 2\pi rh \\
 &= 2 \times \frac{22}{7} \times 3.5 \times 25 \text{ cm}^2 \\
 &= 550 \text{ cm}^2
 \end{aligned}$$

EXERCISE SET 14.2

- Two cylinders, A and B, are given. The radius of cylinder B is twice that of cylinder A, and the height of cylinder B is half that of cylinder A. Find the ratio of the curved surface area of A to the curved surface area of B. Also find the ratio of the volume of A to the volume of B.
- The radii of two cylinders are in the ratio 2:3, and their heights are in the ratio 3:2. Find (a) the ratio of their volumes, and (b) the ratio of their curved surface areas.
- The edge of a cube measures r cm. The largest possible right circular cylinder is cut out of the cube. What do you think is the volume of the cylinder (in cm^3)?
- *4. The radius of the base of a cylinder is increased by 10%. At the same time, the height of the cylinder is decreased by $x\%$. Given that the volume of the cylinder remains unchanged, find the value of x .
5. A solid metallic cube of side 12 cm is melted and recast into solid cylindrical rods, each having radius 2 cm and height 12 cm. Find:
 - (i) the volume of the cube,

- (ii) the volume of one cylindrical rod,
- (iii) the approximate number of complete cylindrical rods that can be formed. $\left(\pi \approx \frac{22}{7}\right)$

14.3 CONES



Fig. 14.7A



Fig. 14.7B

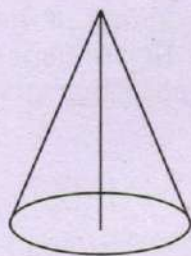
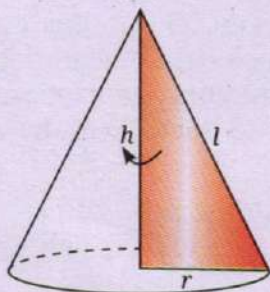


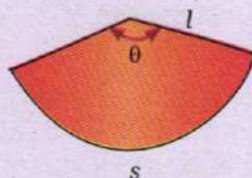
Fig. 14.7C

A right circular cone is generated when a right triangle is rotated about one of its legs. The relationship among the slant length l , the height h , the radius r of the base is

$$l^2 = h^2 + r^2 \text{ (using the Baudhāyana–Pythagoras theorem)}$$



Right circular cone



Unrolled cone

Fig. 14.8

Imagine cutting the cone along a slant edge and 'opening up' or unrolling the surface. The unrolled surface has the shape of a sector of a circle. To create the cone again, bring the two straight edges together. You can use the cutout of the net of a cone given at the end of the textbook to make a cone.

What is the curved surface area of a cone?

The formula for surface area becomes clear if we imagine cutting the cone along a slant edge, unrolling the surface, and cutting it up into

a large number of very thin sectors, each of which can be regarded as a narrow isosceles triangle of height l (Fig. 14.9).

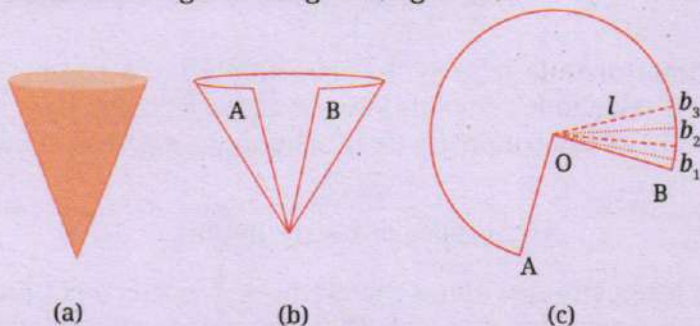


Fig. 14.9

From Fig. 14.9, it follows that the curved surface area (CSA) of the cone is equal to

$$\begin{aligned} \text{CSA} &= \frac{1}{2}b_1l + \frac{1}{2}b_2l + \cdots + \frac{1}{2}b_nl = \frac{1}{2}(b_1 + b_2 + \cdots + b_n)l \\ &= \frac{1}{2} \times \text{circumference of the base} \times l \\ &= \frac{1}{2} (2\pi r)l = \pi rl \end{aligned}$$

Here is another way of working out the formula for the curved surface area. We have, from the formula for area of a sector of a circle,

$$\text{CSA} = \pi l^2 \times \frac{\theta^\circ}{360^\circ} = \pi l \times l \times \frac{\theta^\circ}{360^\circ}$$

Let us try to express this formula in terms of r and l . For this, note that from the cone, we have,

$$\text{arc length} = s = 2\pi r,$$

and from the sector, we have

$$\text{arc length} = s = \frac{\theta^\circ}{360^\circ} \times 2\pi l.$$

Equating these two, we get

$$\frac{\theta^\circ}{360^\circ} = \frac{r}{l}.$$

From this, we see that the curved surface area is

$$\text{CSA} = \pi rl,$$

the same formula as earlier. The formula for the total surface area (TSA) (curved portion plus the flat base) is $\text{TSA} = \pi rl + \pi r^2$.

What is the volume of a cone? It turns out that

$$V = \frac{1}{3} \pi r^2 h.$$

The volume formula tells us that the volume of a cone is exactly $\frac{1}{3}$ the volume of a cylinder formed on the same circular base and with the same height as the cone. The formula may be expressed in different ways, e.g.,

$$V = \frac{1}{3} \times \text{area of base} \times \text{height}.$$

Students often wonder about the factor of $\frac{1}{3}$ in this formula. Why $\frac{1}{3}$? To explain the reason for this factor requires more advanced methods, e.g., calculus, which you will study when you are in Grade 12.

But it is possible to *experimentally* verify this relationship. Let us start with a right circular cylindrical tin. Using paper, we make a right circular cone with the same base radius and the same height as the cylinder (see Fig. 14.10).

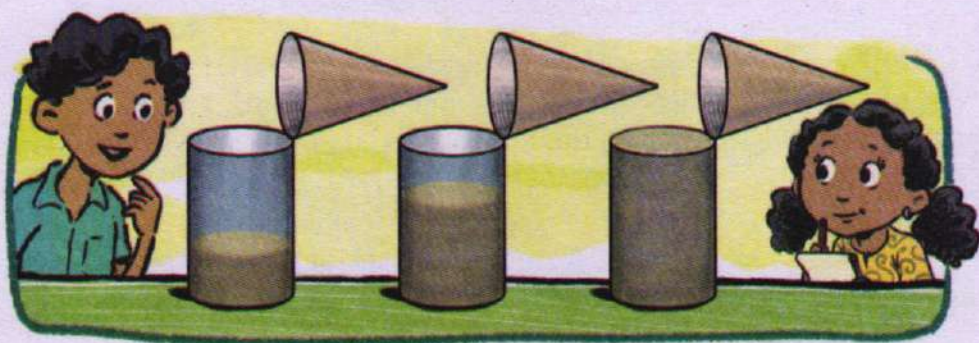


Fig. 14.10

We are now going to perform an experiment to find the volume of the cone. We need some clean fine-grained salt for this purpose.

Fill the cone to the brim with salt. Empty it into the cylinder. We find that it fills up only part of the cylinder. Fill the cone with salt to the brim again and empty it into the cylinder. We find that the cylinder is still not full. When we fill the cone with salt for the third time and empty it yet again into the cylinder, we find that the cylinder is full to the brim.

We can therefore conclude that the volume of the cylinder is three times the volume of the cone. That is, given a right circular cylinder and

a right circular cone of the same base radius and the same height, the cone has one third the volume of the cylinder. This gives an experimental verification of the formula given above.

Here is another hands-on activity that uses modelling clay (earlier this used to be called 'plasticine'). Carefully mold the clay into a solid cylinder and measure its radius and height. Then reshape the same portion of clay into identical cones, each having the same radius and the same height as the cylinder. You will find that you are able to make exactly three such cones. This brings out the relationship between the volume of the cylinder and cone quite visibly

$$V = \frac{1}{3} \times \text{area of base} \times \text{height}.$$

The formula for the volume of a cone was first found by the Greek mathematician Archimedes (c. 225 BCE). He also derived formulas for the surface area and volume of a sphere (i.e., ball), which we will study in later sections.

Example 3: A right triangle ABC with sides 5 cm, 12 cm and 13 cm is rotated through 360° about the side with length 12 cm. Find the volume of the solid so obtained.

The triangle has sides 5 cm, 12 cm, 13 cm.

Since $5^2 + 12^2 = 25 + 144 = 169 = 13^2$, the triangle is a right-angled triangle with perpendicular sides 5 cm and 12 cm and hypotenuse 13 cm.

The triangle is rotated about the side with length 12 cm. When a right-angled triangle rotates about one of its perpendicular sides, it forms a right circular cone. Here, height of cone $h = 12$ cm, radius of cone $r = 5$ cm.

$$\text{Hence, volume of the cone (V)} = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi \times 5^2 \times 12 = 100 \pi \text{ cm}^3.$$

EXERCISE SET 14.3

For π , use one of the following approximations: $\pi \approx \frac{22}{7}$ or $\pi \approx 3.14$

1. Find the total surface area of a cone, if its slant height is 21 m and diameter of its base is 24 m.
2. Find the curved surface area of a right circular cone whose slant height is 10 cm and base radius is 7 cm.

3. The height of a cone is 16 cm, and its base radius is 12 cm. Find the curved surface area and the total surface area of the cone.
4. A cone has a height of 15 cm. If its volume is 1570 cm^3 , find the radius of the base.
5. The curved surface area of a cone is 308 cm^2 and its slant height is 14 cm. Find (i) the radius of the base, (ii) the total surface area of the cone.
6. A joker's cap is in the form of a right circular cone of base radius 7 cm and height 24 cm. Find the area of the sheet required to make 10 such caps.
- *7. What length of tarpaulin 3 m wide is required to make a conical tent of height 8 m and base radius 6 m? Assume that the extra length of material that is required for stitching margins and wastage in cutting is 20 cm.
8. A right triangle with sides 6 cm, 8 cm and 10 cm is rotated through 360° about the side of 8 cm. Find the volume and the curved surface area of the solid so formed.
- *9. Suppose you have a cup in the shape of a right circular cone. Fill it with water to half the depth of the cone.
What fraction of the volume of the cup is occupied by the water?



14.4 PYRAMIDAL SHAPES

A pyramidal shape is of the type shown in the following figures.



Fig. 14.11A

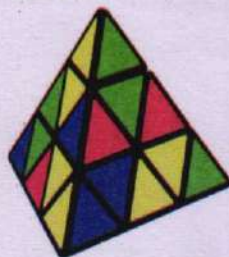


Fig. 14.11B

Its base may be a triangle, a square or indeed any closed shape. All points in the base are connected to a single point (the *apex*) which lies in a different plane. You can make the pyramids by using the cutouts of nets given at the end of the textbook.

The famous pyramids in Egypt have square bases. The puzzle in Fig. 14.11B (made by the Hungarian architect Erno Rubik) has a triangular base.

Some mountains have pyramidal peaks (formed by glacier erosion over many millennia). In India, there is the Shivling Peak in the state of Uttarakhand. An example from the Alps is the Matterhorn.



Shivling Peak (Uttarakhand)



The Matterhorn (Switzerland)

The volume V of a pyramidal shape is found by using the formula

$$V = \frac{1}{3} \times \text{area of base} \times \text{height}.$$

The same formula is applicable whether the base is triangular, square or any other shaped flat surface.

Note that the formula is the same as the one used for the volume of a cone. Indeed, we may think of a cone as a pyramid with a circular base.

The surface area of a pyramidal shape is found by adding up the areas of all the polygonal faces. So, we do not need a separate formula for the surface area.

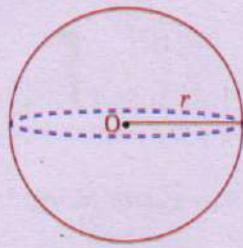
14.5 SPHERES AND HEMISPHERES



Volleyball



Football (Soccer)



Sphere

Lastly, we consider a familiar object in daily life: a ball or a sphere.

This is the three-dimensional (3D) equivalent of a circle. Whereas a circle is the collection of all points **in a plane** that are at some given distance from some given point, a sphere is the collection of all points in 3D space that are at some given distance r (the radius of the sphere) from a given point O , the centre of the sphere. If we rotate the circle about any diameter, we obtain a hollow sphere.



Archimedes showed that the surface area A of a sphere with radius r is equal to $4\pi r^2$. He did this by showing that the surface area of a sphere is exactly equal to the curved surface area of a cylinder with radius r and height $h = 2r$ that fits tightly over the sphere, Fig. 14.12.

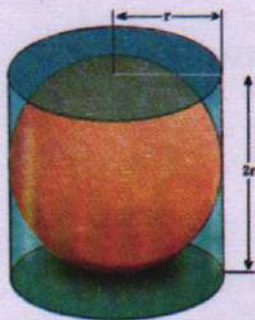


Fig. 14.12

The curved surface area of such a cylinder is equal to

$$2\pi rh = 2\pi r \times 2r = 4\pi r^2,$$

This is the formula obtained by Archimedes.



Fig. 14.13: Rectangular map projection

Source: https://en.wikipedia.org/wiki/Equirectangular_projection#/media/File:Equirectangular_projection_SW.jpg

In passing, we note that this result is used in cartography (map-making), when making a rectangular map projection of the earth. Imagine a cylinder tightly wrapped around the earth, touching it at the equator, as described above. By drawing radial lines from the central north pole-south pole axis, we project horizontally from the earth's surface to the surface of the cylinder. Then, we unwrap the cylinder and obtain the desired map. The resulting world map is rectangular in shape which is easy and convenient to use. But it suffers from an important defect: there is great distortion in shapes at high latitudes (close to the poles). For locations close to the equator, the distortion is negligible.

Archimedes also found the formula for the volume V of a sphere:

$$V = \frac{4}{3} \pi r^3.$$

You will study the derivations of these formulas in later grades, using calculus.

But you may be able to find your own derivation of the formula by connecting it with the formula for volume of a cone; namely, by writing it as

$$V = \frac{1}{3} \times (4\pi r^2) \times r,$$

i.e., Volume of sphere = $\frac{1}{3} \times$ surface area of sphere $\times$ radius of sphere.

Try to work out the connecting links on your own.

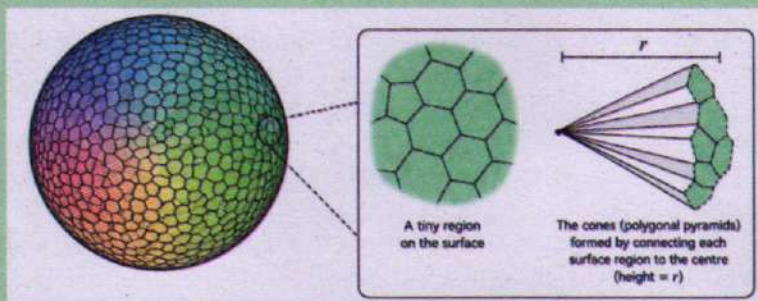
*Visualising Volume Formula

Here is a way to see the link between the formulas for volume of a cone, surface area of a sphere and volume of a sphere. Recall that

$$\text{Volume of cone} = \frac{1}{3} \text{ area of base} \times \text{height of cone.}$$

Imagine that the entire surface of the sphere is divided into tiny regions. The shapes of these regions do not matter; what matters is that they are very tiny. Now connect all these regions to the centre of the sphere. This creates a large number of thin cones. They all have the same height, namely r . The volume of the sphere is equal to the sum of the volumes of all the cones. Using the fact that the cones all have the same height, we may write:

$$\begin{aligned} \text{Volume of sphere} &= \frac{r}{3} \times \text{sum of base areas of all the thin cones} \\ &= \frac{r}{3} \times \text{surface area of sphere} = \frac{r}{3} \times 4\pi r^2 = \frac{4}{3} \pi r^3. \end{aligned}$$



14.5.1 Hands-on Activity to Verify the Formula for Surface Area of a Sphere

Have you ever played with a top? You must know how a string is wound around it. Take a rubber ball and drive a nail into it. Using the nail for support, wind a string tightly around the ball. When you have reached the 'fullest' part of the ball, use pins to keep the string in place, and continue to wind the string around the remaining part of the ball, till you have completely



around the remaining part of the ball, till you have completely

covered it; see Fig. 14.14. Mark the starting and finishing points on the string, and unwind the string from the surface of the ball. Now, measure the diameter of the ball and get its radius. On a sheet of paper, draw four circles with a radius equal to the radius of the ball. Fill the circles with the string you had wound around the ball (as shown in the figure).

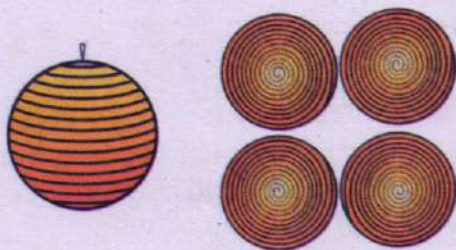


Fig. 14.14

The string, which had completely covered the surface of the sphere, has now been used to fill the four circles, all with the same radius as the sphere. So, the surface area of a sphere of radius r is equal to 4 times the area of a circle of radius r , i.e., equal to $4\pi r^2$. Hence, surface area of a sphere = $4\pi r^2$ where r is the radius of the sphere. This experimentally verifies the stated formula.

14.5.2 Hemisphere

A hemisphere is a three-dimensional solid formed when a sphere is cut into two equal halves by a plane passing through its centre. The flat circular region formed after cutting the sphere is called the *base* of the hemisphere. A hemisphere has one curved surface and one circular base. See Fig. 14.15.



Fig. 14.15: Hemispheres

You may also recall encountering the term hemisphere in the context of the Northern Hemispheres and the Southern Hemispheres in Geography.

Examples: Bowls, serving dishes, dome-shaped covers, planetariums and igloos.

Let the radius of the sphere be r . The surface area of the complete sphere is $4\pi r^2$. Therefore, the curved surface area of a hemisphere, which is half of the sphere, is $2\pi r^2$.

The total surface area (TSA) includes the curved surface and circular base. The area of the circular base = πr^2 .

$$\text{Hence TSA} = 2\pi r^2 + \pi r^2 = 3\pi r^2.$$

$$\text{Volume of the hemisphere} = \frac{2}{3}\pi r^3.$$

Example 4: A hemispherical bowl has a radius of 3.5 cm. What would be the volume of water it would contain?

The volume of water the bowl can contain

$$\begin{aligned} &= \frac{2}{3}\pi r^3 \\ &= \frac{2}{3} \times \frac{22}{7} \times 3.5 \times 3.5 \times 3.5 \text{ cm}^3 = 89.8 \text{ cm}^3. \end{aligned}$$

EXERCISE SET 14.4

1. A ball bearing has a radius of 0.7 cm. Find its surface area.
- *2. Two solid spheres made of the same metal have weights 5920 g and 740 g. Determine the radius of the larger sphere, if the diameter of the smaller one is 5 cm.
3. The diameter of the moon is approximately one fourth the diameter of the earth. Given that the moon and earth are both roughly spherical, find the ratio of their surface areas.
4. Find (i) the curved surface area and (ii) the total surface area of a hemisphere of radius 21 cm.
5. Metal spheres, each of radius 2 cm, are packed into a rectangular box of internal dimensions 16 cm $\times$ 8 cm $\times$ 8 cm. When 16 spheres are packed, the box is filled with preservative liquid. Find the volume of this liquid. Round your answer to the nearest integer.
6. The radius of a sphere is increased by 10%. Show that the volume increases by approximately 33.1%.

- *7. The radius of a sphere is increased by $x\%$. The volume of the sphere increases by 72.8%. Find the value of x .
8. The hemispherical dome of a building needs to be painted (see Fig. 14.16). If the circumference of the base of the dome is 35.2 m, find the cost of painting it, given the cost of painting is ₹ 10 per 100 cm^2 .

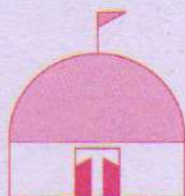


Fig. 14.16

14.6 AREAS AND VOLUMES AROUND US

We learned about different 3D solids. Identify and list objects or structures around you that are cubical, cuboidal, cylindrical, conical, spherical, or pyramidal in shape, or closely resemble these shapes.

Activity: Find the approximate volumes of a gas cylinder, a bucket, any cylindrical vessel, a cupboard, a water tank that you can find near you (use $1000 \text{ cm}^3 = 1 \text{ litre}$).

14.6.1 Guesstimates: Did You Ever Wonder?

Guesstimate questions are problems in which you make a reasonable estimate for a quantity when exact data are not available. Instead of finding a perfectly accurate answer, you use assumptions, approximations, logic, and everyday knowledge to model a given situation and arrive at a sensible estimate. Do you remember solving such questions in earlier grades?

Example 5: Lallan's *golgappa* (a small hollow ball made of wheat) stall has a cylindrical vessel with *pani* (spiced water) in it. The vessel is of radius 10 cm and it has *pani* up to a height of 50 cm. Estimate the number of *golgappas* he can serve assuming all *golgappas* are spheres. Before calculating, make a guess.

Can you guess what number this could be—is it in the hundreds, thousands, or more?

The number of *golgappas* that can be served is

$$\frac{\text{Total } \textit{pani} \text{ in the vessel}}{\text{Volume of } \textit{pani} \text{ in each } \textit{golgappa}}.$$



The amount of *paani* in the cylindrical vessel is given by $V = \text{area of base} \times \text{height}$.

$$V = \pi (10)(10) \times 50 = 5000\pi \text{ cm}^3.$$

To estimate the number of *golgappas*, two students went about in different ways:

Student 1's calculation

Estimated radius of an average *golgappa* is 1.5 cm.

$$\begin{aligned} \text{Volume of a } golgappa &= \frac{4}{3} \times \pi (1.5)^3 \\ &\approx 14.14 \text{ cm}^3. \end{aligned}$$

(neglecting the thickness of the *golgappa*)

Assuming each *golgappa* is filled with 60% *pani* on an average, the volume of *pani* in a *golgappa* = $14.14 \times 0.6 \approx 8.5 \text{ cm}^3$.

The number of *golgappas* that can be served is = $\frac{5000\pi}{8.5} \approx 1848$.

Lallan can serve about 1850 *golgappas* with the *pani* he has.

Student 2's calculation

Estimated radius of an average *golgappa* is 1.75 cm.

$$\begin{aligned} \text{Volume of a } golgappa &= \frac{4}{3} \times \pi (1.75)^3 \\ &\approx 22.43 \text{ cm}^3. \end{aligned}$$

(neglecting the thickness of the *golgappa*)

Assuming each *golgappa* is filled with 80% *pani* on an average, the volume of *pani* in a *golgappa* = $22.43 \times 0.8 \approx 18 \text{ cm}^3$.

The number of *golgappas* that can be served is = $\frac{5000\pi}{18} \approx 873$.

Lallan can serve about 870 *golgappas* with the *pani* he has.

We see that the final answer depends upon the assumptions, estimations, and approximations done in the process.

Example 6: Anirban saw a few tennis balls lying around. He wondered, "How many tennis balls can fit in an empty classroom?". The classroom is of dimensions 30 ft $\times$ 30 ft $\times$ 15 ft. Estimate how many tennis balls of radius 3 cm fit in this classroom. Before calculating, make a guess.

The volume of the tennis ball is $\frac{4}{3} \times \pi \times 3^3 \approx 113 \text{ cm}^3$.



The volume of the classroom is

$$30 \times 30 \times 15 = 13,500 \text{ ft}^3 \approx 37,80,00,000 \text{ cm}^3.$$

Can we find the number of tennis balls that can fit in the classroom by dividing the volume of the classroom by the volume of a tennis ball?

$$1 \text{ ft} = 30.48 \text{ cm}$$

$$1 \text{ ft}^3 \approx 28317 \text{ cm}^3$$

$$\frac{37,80,00,000}{113} \approx 33,45,132.$$

$$1 \text{ ft} \approx 30 \text{ cm}$$

$$1 \text{ ft}^3 \approx 28000 \text{ cm}^3$$

No, the balls cannot fit in the room without leaving gaps in between. Hence this operation will only give an upper bound value, i.e., we can say for sure that we cannot fit more balls than this.

Accounting for these gaps can give a better estimate.

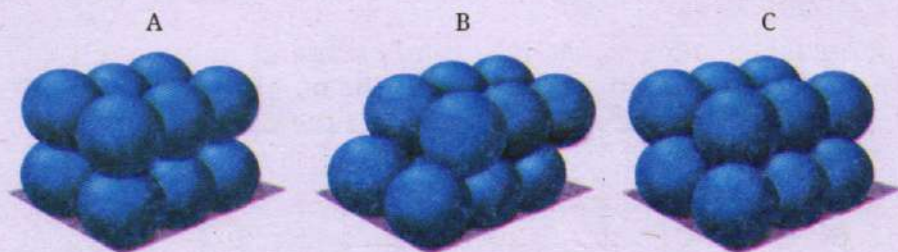


Fig. 14.17: These are some of the ordered ways of packing spheres in a cuboidal structure

Using the first configuration: Along each side of the room, the number of balls that can be placed side-by-side along the length is $\frac{\text{dimension of the room}}{\text{diameter of the ball}} \approx \frac{900}{6} = 150$. The same number can be placed along the width. Along the height, it is 75 balls.

Therefore, $150 \times 150 \times 75 = 16,87,500$ denotes the approximate number of balls that can fill the room.

Using the other configurations, we might be able to fit more balls because of higher packing efficiency.

Make an estimate based on the second and third configurations.

Note: This is why spheres are not used as basic units to measure volumes as they leave gaps unlike cubes.

As we have seen in guesstimate problems, different people may get different answers, but the reasoning they use matters the most. For these problems, you can start with a quick intuitive guess, model the situation by identifying the relationships among the quantities, make reasonable assumptions, and then calculate an answer by making suitable approximations. Look back to see how close your intuitive guess is to your estimate.

END-OF-CHAPTER EXERCISES

For guesstimate problems, make a guess first before solving.

1. Estimate how many scoops of ice cream can be obtained from a cuboidal container of dimensions $10\text{ cm} \times 15\text{ cm} \times 20\text{ cm}$.



2. A cube of integer side length a is made of unit cubes. Write an expression giving the number of unit cubes to be added to make a cube of side length $a + 1$.

- *3. Solve the following:

- (i) Could a person drink enough water in a lifetime to fill an entire room the size of your classroom?

- (ii) Estimate the number of bricks used to build the walls of your classroom.



4. You are given two options: (A) one chocolate cube of side 50 mm , (B) hundred chocolate cubes of side 10 mm . Which option gives you more chocolate?

5. If all the people in the world are crowded together in one location, how much area would it cover?

6. A school provides milk to its students in cylindrical glasses, each with a diameter of 7 cm . If each glass is filled with milk to a height of 12 cm , how many litres of milk are needed to serve 1600 students? Use $1000\text{ cm}^3 = 1\text{ litre}$.



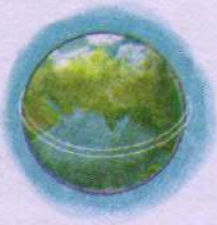
7. The surface area of a sphere of radius 5 cm is five times the area of the curved surface of a cone of radius 4 cm . Find the height and the volume of the cone.

8. Take Earth to be a perfect sphere with radius 6370 km. Take Jupiter to be a perfect sphere with radius 69,900 km. Take the Sun to be a perfect sphere with radius 6,95,700 km. Compute approximately:

(i) the ratio of the volume of Jupiter to the volume of the Earth;
 (ii) the ratio of the volume of the Sun to the volume of the Earth.
 (Calculator can be used.)

9. Show that the volume of a sphere is equal to $\frac{2}{3}$ of the volume of the smallest cylinder which encloses it.

- *10. Suppose the Earth is perfectly spherical. A string is wrapped tightly around the equator of the Earth. Another string, 1 metre longer, is placed around the Earth so that it forms a larger circle, staying the same distance above the ground everywhere.



How high above the ground is the second string?

Repeat this for: (i) the Moon (ii) Jupiter (iii) a volleyball. Are you surprised by the three answers? Why or why not?

- *11. We have a cylinder with a base radius of r cm and height h cm. A square pyramid is fitted inside it. The square base of the pyramid lies on the base of the cylinder, its corners on the boundary of the cylinder. The apex of the pyramid lies on the top of the cylinder. Find the ratio of the volume of this pyramid to the volume of the cylinder.

12. What is the change in volume when:

(i) the length of a cuboid with dimensions l, w, h is increased by 1 unit?

- (a) 1 cubic unit
 (b) $(lwh + 1)$ cubic unit
 (c) wh cubic units
 (d) lw cubic units
 (e) lh cubic units

(ii) the radius of a cylinder with dimensions r, h is increased by 1 unit?

- (a) 1 cubic unit
 (b) $\pi r^2 h$ cubic units
 (c) πr^2 cubic units
 (d) $2\pi rh + 2\pi h$ cubic units
 (e) $2\pi rh + \pi h$ cubic units

*(iii) the radius of a sphere is decreased by 1 unit?

- *13. Given a cube with volume V , express its total surface area S in terms of V .

- *14. Given a cube with total surface area S , express its volume V in terms of S .
15. A cylindrical glass of height 25 cm and radius 4 cm has water up to a height of 16 cm. A crow wants to drink water from this glass. The water must be at a height of 20 cm for the crow to reach it. There are some marbles lying around. How many marbles of radius 1 cm should the crow drop into the glass to make the water reach the required height?



16. Find the volume of ink in a new ball point pen. Take the necessary measurements and make approximations, as needed.

17. Solve:

- (i) A ball of chapati dough of radius 6 cm is prepared. Estimate how many chapatis can be made from it?
- * (ii) Cut a coconut/muskmelon in half and find out the approximate volume of edible coconut flesh/fruit by taking the necessary measurements.

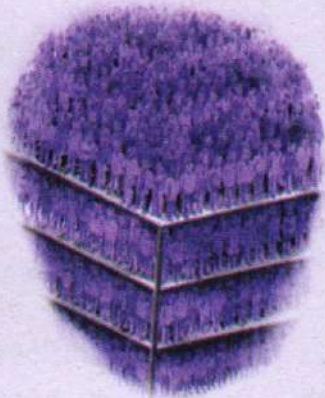


18. Looking at the *Ganita Manjari*, Grade 9, Part 2 textbook, Sheela wonders:

- (i) If all the pages of this textbook were laid out side-by-side on the floor would they cover the entire classroom floor?
- (ii) What is the maximum number of textbooks that can fit in an empty storeroom of dimensions 15 ft $\times$ 20 ft $\times$ 30 ft?

- *19. If the entire human population decided to climb into one giant cube, how long would the side have to be?

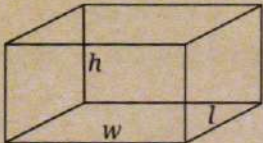
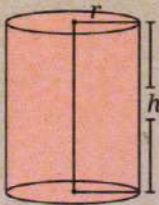
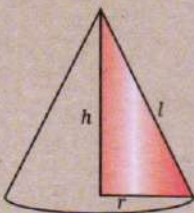
- *20. The Earth's surface has an estimated volume of 1.38 billion km^3 of water. Suppose the Earth is a perfect sphere and all this water forms a uniform layer completely covering the Earth's surface, like a thin water bubble. Estimate the thickness of this water layer. The Earth's radius is ~ 6371 km.

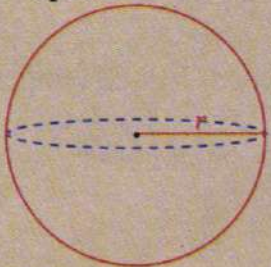
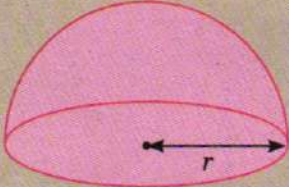


- (i) Write an expression that gives the thickness of this water layer.
 (ii) Simplify the expression in (i) using a calculator.
- *21. Give the dimension of a cuboid whose volume is halved when its surface area is doubled.
22. **Project:** Find the volume of your house making necessary approximations. Present how you solved it.

CHAPTER SUMMARY

In this chapter, we have studied the following formulas for the surface areas and volumes of various fundamental shapes.

3D Shape	Surface Area	Volume
1. Cuboid 	$2(wl + hw + lh)$	whl
2. Right Circular Cylinder 	Curved surface area: $2\pi rh$ Total Surface Area of a cylinder covered at one end: $2\pi rh + \pi r^2 = \pi r(2h + r)$ Total Surface Area of a cylinder covered at both ends: $2\pi rh + 2\pi r^2 = 2\pi r(h + r)$	$\pi r^2 h$
3. Right Circular Cone 	Curved surface area: πrl Total surface area: $\pi r(l + r)$	$\frac{1}{3} \pi r^2 h$

4. Sphere  A diagram of a sphere with a red outline. A horizontal dashed blue line represents the equator. A red radius line is drawn from the center to the right edge of the sphere, labeled with the letter 'r'.	$4\pi r^2$	$\frac{4}{3}\pi r^3$
5. Hemisphere  A diagram of a hemisphere with a pink shaded curved surface and a pink elliptical base. A horizontal double-headed arrow is drawn across the base, labeled with the letter 'r'.	Curved surface area: $2\pi r^2$ Total surface area: $2\pi r^2 + \pi r^2$ $= 3\pi r^2$	$\frac{2}{3}\pi r^3$